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Dynamical Chern–Simons modified gravity, Gödel Universe and variable cosmological constant

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ABSTRACT

We study the condition for the consistency of the Gödel metric with the dynamical Chern–Simons modified gravity. It turns out to be that this compatibility can be achieved only if the cosmological constant is variable in the space.

The problem of breaking of the Lorentz and CPT symmetries is intensively studied now. Among a great number of possible Lorentz-CPT breaking modification of gravity [1], the four-dimensional gravitational Chern–Simons term originally introduced in [2] is certainly the most popular one. The main reasons for it are its relatively simple form, evident gauge symmetry and straightforward analogies with the three-dimensional gravitational Chern–Simons term [3]. A lot of issues treating different aspects of the four-dimensional gravitational Chern–Simons term has been studied, see [4] for a general review. In particular, in [5] it was shown to emerge as a quantum correction. The most important line of studies of this term is actually devoted to searching for the solutions of the equations of motion in the modified theory whose action is a sum of the usual Einstein–Hilbert action and of the gravitational Chern–Simons term. It was shown that a lot of well-known solutions of the usual Einstein equations, in particular, spherically symmetric and cylindrically symmetric ones, such as Schwarzschild, Reissner–Nordstrom and Friedmann–Robertson–Walker metrics resolve the modified Einstein equations as well [6].

However, the Kerr metric failed to solve the modified Einstein equations. To circumvent this difficulty, in [6,9], the concept of the dynamical Chern–Simons coefficient was introduced. Within this formulation, where the function θ multiplying the topological $^*RR$ term in the modified gravity action is considered as a dynamical variable, it was shown that the Kerr metric must be adequately modified to satisfy the new modified equations of motion involving dynamics of θ as well [10]. Therefore, a natural problem emerges – whether other known metrics, or, probably, other parameters of the theory must suffer any modification to solve the modified equations of motion?

In this Letter we are going to consider this problem for the Gödel metric [8] interest to which highly uprose in the recent years [11]. This metric, known initially for the possibility of the closed timelike curves (CTCs), has been successfully embedded into five-dimensional supergravity [12], then it was generalized for higher space–time dimensions [11]. Moreover, it was shown in [11] that there exist brane solutions locally identical to the Gödel metric (see also [13]), thus, the Gödel metric turns out to have an important role in the supergravity context which justifies interest to its study. Earlier, the compatibility of this metric with modified Einstein equations has been shown in the case of the usual (non-dynamical) Chern–Simons coefficient in [7]. Our result turns out to be highly nontrivial – indeed, we find that this metric solves new modified equations of motion only in the case of variable cosmological constant! This effect can be treated as a justification of the hypotheses of the variable cosmological constant which are very popular in the context of search for a possible explanation of accelerated expansion of the Universe (see f.e. [14]).

The starting point of our consideration is the following action for the Chern–Simons modified gravity:

$$S=\frac{1}{16\pi G}\int d^4x\left[\sqrt{-g}(R-\Lambda)+\frac{l}{64\pi}\theta^*RR-\frac{1}{2}\partial^{\mu}\theta\partial_{\mu}\theta\right]+S_{mat}, \tag{1}$$

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where, unlike [7], the function θ is a dynamical variable [9]. Varying this action with respect to the metric and to the scalar field θ, we obtain the following equations of motion:

$$G_{\mu\nu} + lC_{\mu\nu} = T_{\mu\nu}, \tag{2}$$

$$g^{\mu\nu}\partial_\mu\partial_\nu\theta = -\frac{l}{64\pi}{}^*RR, \tag{3}$$

where $G_{\mu\nu}$ is the Einstein tensor (within this formulation, the contribution from the cosmological term is absorbed into the energy–momentum tensor) and $C_{\mu\nu}$ is the Cotton tensor defined as

$$C^{\mu\nu} = -\frac{1}{2\sqrt{-g}}\left[v_\sigma\epsilon^{\sigma\mu\alpha\beta}D_\alpha R^\nu_\beta + \frac{1}{2}v_{\sigma\tau}\epsilon^{\sigma\nu\alpha\beta}R^{\tau\mu}{}_{\alpha\beta}\right] + (\mu \longleftrightarrow \nu), \tag{4}$$

and ${}^*RR$ is the topological invariant called the Pontryagin term whose explicit form is

$${}^*RR \equiv {}^*R^a{}_b{}^{cd}R^b{}_{acd}, \tag{5}$$

where R^b_{acd} is the Riemann tensor and ${}^*R^a{}_b{}^{cd}$ is the dual Riemann tensor given by

$${}^*R^a{}_b{}^{cd} = \frac{1}{2}\epsilon^{cdef}R^a{}_{bef}. \tag{6}$$

In this Letter we are concentrated on the case of the Gödel metric which is written as [8]

$$ds^2 = a^2\left[dt^2 - dx^2 + \frac{1}{2}e^{2x}dy^2 - dz^2 + 2e^x\,dt\,dy\right], \tag{7}$$

where a is a positive number.

The energy–momentum tensor $T_{\mu\nu}$ in this theory is composed of two terms:

$$T_{\mu\nu} = T^m_{\mu\nu} + T^\theta_{\mu\nu}, \tag{8}$$

where $T^m_{\mu\nu}$ is the energy–momentum tensor of an usual matter (including the contribution from the cosmological term). We suggest that this energy–momentum tensor is the same one that yields Gödel solution in the simple Einstein gravity [8] as well as in the Chern–Simons modified gravity with the external θ parameter [7], that is, $T^m_{\mu\nu} = 8\pi\rho u_\mu u_\nu + \Lambda g_{\mu\nu}$, where u is a unit time-like vector whose explicit contravariant components look like $u^\mu = (\frac{1}{a}, 0, 0, 0)$ and the corresponding covariant components are $u_\mu = (a, 0, ae^x, 0)$. Further, the energy–momentum tensor for the θ field looks like

$$T^\theta_{\mu\nu} = (\partial_\mu\theta)(\partial_\nu\theta) - \frac{1}{2}g_{\mu\nu}\big(\partial^\lambda\theta\big)(\partial_\lambda\theta), \tag{9}$$

that is, it reproduces the form of the usual energy–momentum tensor for the scalar field in a curved space.

We have showed in [7], that the choice $\theta = \theta(x, y)$ annihilates the Cotton tensor. Let us verify a compatibility of the Gödel metric with the equations of motion (3) just in this case simplifying all expressions drastically. For the θ dependent only on x and y, the non-zero components of the $T^\theta_{\mu\nu}$ look like

$$T^\theta_{00} = \frac{1}{2}(\partial_1\theta)^2 + e^{-2x}(\partial_2\theta)^2,$$

$$T^\theta_{02} = \frac{1}{2}e^x(\partial_1\theta)^2 + e^{-x}(\partial_2\theta)^2,$$

$$T^\theta_{11} = \frac{1}{2}(\partial_1\theta)^2 - e^{-2x}(\partial_2\theta)^2,$$

$$T^\theta_{12} = (\partial_1\theta)(\partial_2\theta),$$

$$T^\theta_{22} = \frac{1}{4}e^{2x}(\partial_1\theta)^2 + \frac{1}{2}(\partial_2\theta)^2, \tag{10}$$

$$T^\theta_{33} = -\frac{1}{2}(\partial_1\theta)^2 - e^{-2x}(\partial_2\theta)^2. \tag{11}$$

Taking into account the expression for the Einstein tensor from [8], that is,

$$G_{00} = \frac{1}{2}, \qquad G_{11} = \frac{1}{2}, \qquad G_{20} = \frac{1}{2}e^x,$$
$$G_{22} = \frac{3}{4}e^{2x}, \qquad G_{33} = \frac{1}{2}, \tag{12}$$

one can write down the nontrivial components of the Einstein equations, 00, 11, 22 and 33 respectively:

$$\frac{1}{2} = 8\pi\rho a^2 + \Lambda a^2 + \left[\frac{1}{2}(\partial_1\theta)^2 + e^{-2x}(\partial_2\theta)^2\right], \tag{13}$$

$$\frac{1}{2} = -\Lambda a^2 + \left[\frac{1}{2}(\partial_1\theta)^2 - e^{-2x}(\partial_2\theta)^2\right], \tag{14}$$

$$\frac{3}{2} = 16\pi\rho a^2 + \Lambda a^2 + \left[\frac{1}{2}(\partial_1\theta)^2 + e^{-2x}(\partial_2\theta)^2\right], \tag{15}$$

$$\frac{1}{2} = -\Lambda a^2 - \left[\frac{1}{2}(\partial_1\theta)^2 + e^{-2x}(\partial_2\theta)^2\right]. \tag{16}$$

The equations for the components 00 and 02 turn out to coincide identically. Beside of this, the component 12 of modified Einstein equations is

$$(\partial_1\theta)(\partial_2\theta) = 0. \tag{17}$$

One can verify that if one would try to introduce the potential $V(\theta)$ for the θ field, where all components ij of the Einstein equations (13)–(16) acquire additive terms $-g_{ij}V(\theta)$, the system (13)–(16) would be consistent only if $V(\theta) = 0$, therefore, the Gödel metric is compatible only with the trivial potential, at least if we want to preserve the condition of vanishing the Cotton tensor.

A straightforward inspection of these components allows to conclude that solutions of the equations for the components 00, 02, 22, 33 can be formally written as

$$8\pi\rho = \frac{1}{a^2}; \qquad \Lambda = -\frac{1}{2a^2} - \frac{\Theta}{a^2}, \tag{18}$$

where

$$\Theta = \frac{1}{2}(\partial_1\theta)^2 + e^{-2x}(\partial_2\theta)^2. \tag{19}$$

Therefore we conclude that these solutions correspond to the case when the cosmological constant is not a constant more but an external field dependent on the space–time coordinates which can be considered as a constant only approximately. However, this does not modify derivation of the corresponding term of the Einstein equations since the Λ (and, as a consequence, Θ) does not depend on the metric tensor, hence the formal structure of the Einstein equations is the same with only difference is in the fact that the cosmological "constant" Λ is now not a constant but a fixed function which does not depend on the metric tensor (the possibility of the spatial dependence of Λ has been argued in a similar way in [18]).

It is easy to see that in the limit $\Theta \to 0$ we recover the usual solution for the Gödel universe [8]. However, to proceed further, one must take into account that the condition (18) does not solve Eq. (14) for the component 11. Returning again to Eq. (17) we find that one must choose either $(\partial_1\theta) = 0$ or $(\partial_2\theta) = 0$. If, for example,