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Pre-inflationary homogenization of scalar field cosmologies

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ABSTRACT

We consider the evolution of covariant and gauge invariant linear density perturbations of scalar field cosmologies using a dynamical systems' approach. We find conditions for which the perturbations decay in time, so that the spacetime approaches a homogeneous solution which inflates, for quadratic and exponential potentials. This pre-inflationary homogenization is found to be stable in the potentials' parameter spaces. Furthermore, in each case, we determine the minimum size of the resultant homogeneous patch and show that, for quadratic potentials, the resulting inflationary solutions include those with the necessary number of e-folds.

1. Introduction

In the inflationary scenario, an early period of accelerated expansion is usually assumed to explain the present large-scale homogeneity and spatial flatness of the universe. However, it has been shown, under fairly general assumptions such as the weak energy conditions, that inflationary models require pre-existing homogeneity over a horizon volume [1–3].

Therefore, an important question is under what conditions can a universe with initial small inhomogeneities approach a homogeneous state which develops inflation. Several past works considered this problem either by using numerical approximations or particular exact solutions to the Einstein field equations (EFEs) with scalar fields.

Goldwirth and Piran [4–7] considered inhomogeneous scalar fields on Friedmann–Lemaitre–Robertson–Walker (FLRW) backgrounds in a so-called effective density approximation and using numerical schemes in spherical symmetry taking into account the backreaction in the metric. They concluded that new inflation requires homogeneity over a region of several horizon sizes and that chaotic inflation requires a sufficiently high average value of the scalar field over several horizon sizes. Other authors arrived to similar conclusions using different approaches, see e.g. [3,8]. A particularly interesting result along these lines is due to Deruelle and Goldwirth [9] who considered a semi-numerical analysis, for inhomogeneous quasi-isotropic universes using a long wavelength iterative scheme, and found sufficient conditions for the onset of inflation by limiting the degree of inhomogeneity in their models. More recently, Sakai [10] used numerical approximations in a spherically symmetry model to find an example of topological inflation which allows higher degrees of inhomogeneity over horizon sizes, provided the vacuum expectation value is large enough.

Exact anisotropic and inhomogeneous symmetric spacetimes have also been considered in the past to address this problem. Burd and Barrow [11] took scalar fields with exponential potentials on spatially homogeneous but anisotropic G_3 backgrounds and found that if inflation occurs, then isotropy is always reached (see also [12,13] and references therein). In turn, Ibañez et al. [14, 15] have looked at inhomogeneous G_2 exact scalar field solutions with exponential potentials and compared in each case: asymptotic isotropization, approach to inflation and the existence of inhomogeneities. They find, in particular, classes of models which do not isotropize. Also connected to this problem, we note the recent result of Bolejko and Stoeger [16] who investigated inhomogeneous dust spherical spacetimes and found non-zero measure sets of initial conditions that give rise to spontaneous homogenization of cosmological models.

It would then be important to consider now non-symmetric inhomogeneous models in cosmology. This can be done using a linear perturbative analysis which includes small initial inhomogeneities.

We shall address this question in the context of scalar field cosmologies which have been considered as early universe models,

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see e.g. [17] and references therein. In particular, we take some of most studied inflationary solutions in FLRW spacetimes due to scalar fields with a quadratic potential (e.g. slow-roll inflation), and with an exponential potential (power-law inflation) in order to study the evolution of linear scalar perturbations. We shall use a covariant and gauge invariant perturbative approach, which, by construction, has the advantage of having clear geometrical and physical interpretations in cosmology [18–20]. We shall combine this framework together with a dynamical systems' approach in order to prove the stability of the pre-inflationary homogenization process for quadratic and exponential potentials.

The Letter is organized as follows: In Section 2, we present our perturbative framework, derive the system of perturbation equations for a general scalar field potential and calculate the system's fixed points. Section 3, contains our main results about the stability of the pre-inflationary homogenization for scalar fields with exponential and quadratic potentials, including a phase space analysis of the respective perturbation systems. Section 4 contains a summary of our conclusions.

2. Covariant and gauge invariant linear density perturbations

It is well known that, for linear perturbations of any FLRW background spacetime, scalar perturbation variables $\Delta(x,\tau)$ satisfy a partial differential equation (PDE) of the form

$$\Delta'' + \mathcal{A}(\tau)\Delta' + \mathcal{B}(\tau)\Delta = \frac{D^2}{H^2}\Delta, \tag{1}$$

where $\mathcal{A}(\tau)$ and $\mathcal{B}(\tau)$ depend on the background solution, H is the Hubble function, the prime represents differentiation with respect to conformal time τ and D^2 is the Laplace–Beltrami operator. A common procedure when analyzing cosmological perturbations is to turn the PDE into an ordinary differential equation (ODE) for Δ by using the harmonic decomposition $\Delta = \sum_n \Delta_{(n)} Q_{(n)}$ such that $D^2 Q_{(n)} = -\frac{n^2}{a^2} Q_{(n)}$, where $n \in \mathbb{N}$ is the wave number, a the FLRW scalar factor and $Q'_{(n)} = 0$.

We shall use this procedure and consider a FLRW background with a scalar field ϕ and potential $\mathcal{V}$. Furthermore, we shall use the phase of perturbation variable (see [21,22])

$$\mathcal{U}_{(n)} := \frac{\Delta'_{(n)}}{\Delta_{(n)}} \tag{2}$$

and the following expansion normalized variables (see e.g. [17])

$$\Psi := \frac{\psi}{\sqrt{6}H}, \qquad \Phi := \frac{\sqrt{\mathcal{V}}}{\sqrt{3}H}, \qquad K := -\frac{{}^3R}{6H^2} \tag{3}$$

where $\psi = H\phi'$, and ${}^3R$ is the 3-Ricci scalar. Then, we can show (see [23] for details and also [20]) that Eq. (1) coupled to the FLRW scalar field background evolution equations result in the following system for the unknown state vector $((\Psi,\Phi),\mathcal{U})$:

$$\begin{aligned}
\mathcal{U}'_{(n)} &= -\mathcal{U}^2_{(n)} - \xi(\Psi,\Phi)\mathcal{U}_{(n)} - \zeta(\Psi,\Phi),\\
\Psi' &= 2\Psi^3 - \left(2+\Phi^2\right)\Psi - \sqrt{6}\Phi\frac{d\Phi}{d\phi},\\
\Phi' &= -\Phi^3 + \left(1+2\Psi^2\right)\Phi + \sqrt{6}\Psi\frac{d\Phi}{d\phi}
\end{aligned} \tag{4}$$

subject to the background constraint equation

$$\Psi^2 + \Phi^2 + K = 1 \tag{5}$$

and with

$$\begin{aligned}
\xi(\Psi,\Phi) &= -\left[1 + \frac{12}{\sqrt{6}}\frac{\Phi}{\Psi}\frac{d\Phi}{d\phi}\right],\\
\zeta(\Psi,\Phi) &= -2\left[3\Phi^2 + \frac{6}{\sqrt{6}}\frac{\Phi}{\Psi}\frac{d\Phi}{d\phi}\left(5 + \frac{6}{\sqrt{6}}\frac{\Phi}{\Psi}\frac{d\Phi}{d\phi}\right)\right.\\
&\qquad \left. + 6\Phi\frac{d^2\Phi}{d\phi^2} - \frac{n^2}{2a^2H^2}\right].
\end{aligned}$$

The use of the variable $\mathcal{U}_{(n)}$ makes the analysis of the system's stability quite transparent: If an orbit is asymptotic to an equilibrium point, the perturbation approaches a stationary state either: decaying to zero if $\mathcal{U}_{(n)} < 0$, growing if $\mathcal{U}_{(n)} > 0$ or having a constant value $\mathcal{U}_{(n)} = 0$. If the orbit is asymptotic to a periodic orbit in the cylinder, then the perturbations propagate as waves (see [22] for details).

In order to study the stability of the flat background inflationary solutions, we take $K = 0$, in which case the fixed points $\mathcal{P}$ of the system (4) are given by:

$$\begin{aligned}
\mathcal{P}:\quad & \frac{d\Phi}{d\phi} = \frac{\sqrt{6}}{6}(1-\Phi)(\Phi-4),\\
\mathcal{U}^{\pm}_{(n)}(\mathcal{P}) &= \frac{1}{2}\left(-\xi(\mathcal{P}) \pm \sqrt{\xi^2(\mathcal{P}) - 4\zeta(\mathcal{P})}\right)
\end{aligned} \tag{6}$$

with

$$\begin{aligned}
\xi(\mathcal{P}) &= 6\Phi^2_{\mathcal{P}} - 1,\\
\zeta(\mathcal{P}) &= 24\Phi^2_{\mathcal{P}} - 18\Phi^4_{\mathcal{P}} - 12\Phi_{\mathcal{P}}\left(\frac{d^2\Phi}{d\phi^2}\right)_{\mathcal{P}} + \frac{n^2}{H^2a^2}.
\end{aligned}$$

It is important to note that, since the background evolution equations form an autonomous subsystem, the background fixed points are also fixed points of the system (4). Moreover, we can restrict the phase space analysis to either the state-vector $(\Psi,\mathcal{U}_{(n)})$ or $(\Phi,\mathcal{U}_{(n)})$, since Ψ and Φ are related through the background constraint (5).

3. Pre-inflationary homogenization of scalar field cosmologies

There are many examples of scalar field potentials in cosmology (see e.g. [24,25]). Among these, two of major importance are the quadratic and exponential potentials. The former are physically relevant due to the so-called slow roll regime, after which the physical process of reheating occurs [26]. In turn, exponential potentials can give rise to models which are consistent with observations of the present acceleration of the universe and their theoretical importance arises from scalar-tensor and string theories (see e.g. [27] and references therein).

A dynamical systems approach to study the generality of inflation in FLRW scalar field cosmologies has been extensively used, particularly for exponential potentials. In that case, the spacetime symmetry leads to the decoupling of the Raychaudhuri equation from the evolution equations (for the normalized variables, see [17]), which can then be written as a 2-dimensional dynamical system. In the case of quadratic potentials, there is no such symmetry and the evolution equations form a higher-dimensional system. However, in that case, there are other interesting approaches to the problem (see [28–31,33]). Here, we will make use of the approach in [33], which is reviewed in Section 3.2.

3.1. Exponential potentials

In the case of exponential potentials one has

$$\mathcal{V}(\phi) = \Lambda e^{\lambda\phi}, \qquad \frac{d\Phi}{d\phi} = \frac{\lambda}{2}\Phi \quad \text{and} \quad \frac{d^2\Phi}{d\phi^2} = \left(\frac{\lambda}{2}\right)^2\Phi, \tag{7}$$