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Systematization of basic divergent integrals in perturbation theory and renormalization group functions

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ABSTRACT

We show that to n loop order the divergent content of a Feynman amplitude is spanned by a set of basic (logarithmically divergent) integrals $I_{\log}^{(i)}(\lambda^2)$, $i = 1, 2, \dots, n$, λ being the renormalization group scale, which need not be evaluated. Only the coefficients of the basic divergent integrals are show to determine renormalization group functions. Relations between these coefficients of different loop orders are derived.

1. Introduction

Implicit regularization (IR) is a non-dimensional momentum space framework which has been claimed to be a strong candidate for an invariant regularization suitable to develop perturbation theory in supersymmetric gauge field theories [1–17]. Assuming an implicit regulator in a general (multiloop) Feynman amplitude, a mathematical identity at the level of propagators allows to write the divergent content as basic divergent integrals (BDI) or loop integrals written in terms of one internal momentum only in an unitarity preserving fashion. This is possible because BPHZ subtractions as well as the counterterm method are compatible with IR to arbitrary loop order. An arbitrary scale appears via a regularization-independent identity which relates two logarithmically BDIs by trading a mass parameter m (or an infrared regulator in the propagators) for an arbitrary positive parameter λ ($[\lambda] = M$) plus a function of m/λ. Consequently, λ parametrizes the freedom of separating the divergent content of an amplitude and acts as a renormalization group scale. The key point underlying IR is that neither the (regularization-dependent) BDIs nor their derivatives with respect to λ represented by BDIs need be evaluated. In other words, the BDIs are readily absorbed into renormalization constants whose derivatives with respect λ used to calculate renormalization group functions can also be expressed by BDIs. The advantage of such scheme is that a physical amplitude is written as a finite part plus a set of BDIs say $I_{\log}^{(i)}(\lambda^2)$ and finite surface terms (STs) expressed by volume integrals of a total derivative in momentum space which stem from (finite) differences between $I_{\log}^{(i)}(\lambda^2)$ and $I_{\log}^{(i)\mu_1\mu_2\cdots}(\lambda^2)$ where the latter is a logarithmically divergent integral which contains in the integrand a product of internal momenta carrying Lorentz indices $\mu_1, \mu_2, \dots$. In other words throughout the reduction of the amplitude to loop integrals, $I_{\log}^{(i)\mu_1\mu_2\cdots}(\lambda^2)$ may be written as a product of metric tensors symmetrized in the Lorentz indices times $I_{\log}^{(i)}(\lambda^2)$ plus a surface term.

Such STs are in principle arbitrarily valued. However, it has been shown that setting them to zero ab initio corresponds to both invoking translational invariance of Green's functions and allowing shifts in the integration variable in momentum space [4,5] which in turn is an essential ingredient to demonstrate gauge invariance based on a diagrammatic proof. Therefore STs seem to encode the possible symmetry breakings. Moreover, it has been verified that constraining such surface terms to nought is also sufficient to guarantee that supersymmetry is preserved in the Wess–Zumino model to 3nd-loop order [10] and supergravity to 1-loop order [14]. Notwithstanding it is reasonable to assert that IR is a good candidate to an invariant calculational friendly regularization framework valid in arbitrary loop order. From

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the point of view of algebraic renormalization, STs would be the necessary symmetry restoring counterterms whose expression is known within IR. Then a constrained version of IR (CIR) amounts to setting them to zero from the start and thus constituting an invariant scheme. When physical quantum breakings (anomalies) are expected some care must be exercised: one is able to spot a genuine breaking by letting the STs to be arbitrary so to verify that none consistent set of values for the STs dictated by symmetry requirements fulfill all the essential Ward identities of the underlying model at the same time [7,12]. In [2,5] the rules that define IR to arbitrary loop order are specified.

A renormalization group equation can immediately be written within IR adopting λ as a renormalization group scale and a minimal, mass-independent renormalization scheme in which only the basic divergent integrals are absorbed in the renormalization constants. Hence renormalized Green's function satisfy a kind of Callan–Symanzik equation governed by the scale λ.

The purpose of this contribution is to twofold. Firstly although IR works in arbitrary massive quantum field theories, for massless theories it undergoes a remarkable simplification. Assuming an infrared regulator μ for the propagators, $I^{(i)}_{\log}(\mu^2)$ equals $I^{(i)}_{\log}(\lambda^2)$ $(\lambda \neq 0)$, plus a sum of terms proportional to powers of the logarithm of the ratio μ/λ. We will show in this contribution that for massless theories all the divergencies to arbitrary loop order can be cast as a function of $I^{(i)}_{\log}(\lambda^2)$, according to the definition

$$I^{(i)}_{\log}(\mu^2) = \int_k^{\Lambda} \frac{1}{(k^2-\mu^2)^2} \ln^{(i-1)}\left(-\frac{k^2-\mu^2}{\lambda^2}\right), \tag{1}$$

where $\int_k^{\Lambda} \equiv \int (d^4k)/(2\pi)^4$ and the superscript Λ is a symbol for an implicit regularization. Secondly it is well known that renormalization group functions constitute a testing ground for regularizations because they both encode the symmetry properties of the underlying model which should be preserved by the regularizations and their expansion in perturbation theory contains terms which are universal, i.e. renormalization scheme-independent. While some interesting simplifications take place in dimensional methods, e.g. in an inverse power series in $\epsilon \to 0$ of the coupling constant, beta functions are determined uniquely by the residue of the simple pole on ϵ, it is pertinent to ask what is the counterpart in IR. That is to say, one may wonder how the calculation of renormalization group functions systematizes within a scheme where only basic divergent integrals are claimed to be sufficient to exhibit the ultraviolet properties of a model in a symmetry preserving fashion. The answer to this question is that a general framework for renormalization group functions can be built in which the simplifications of dimensional methods manifest themselves as relations between the coefficients of basic divergent integrals coming from different Feynman graphs that contribute to a given renormalization group function.

We illustrate with the Yukawa model in $3+1$ dimensions to 2nd-loop order which contains a γ_5 matrix and hence the application of dimensional regularization is more involved.

2. General ultraviolet structure of massless theories

The purpose of this section is to show that the ultraviolet content of an amplitude to nth loop order for massless models, considering the definition, is written in terms of $I^{(i)}_{\log}(\lambda^2)$. A general n-loop, l-point amplitude, after space–time and internal group algebra contractions are performed, can always be written as a combination of integrals of the type

$$\int_k^{\Lambda} \frac{k_{\mu_1}k_{\mu_2}\cdots k_{\mu_j}}{(k-p_1)^2\cdots(k-p_l)^2} \mathcal{A}_{n-1}\big(k, p_1, \ldots, p_l, \lambda^2\big), \tag{2}$$

where we have integrated $n-1$ times leaving only k, the most external loop momentum and the p_i's are external momenta. For a massless model suppose that $\mathcal{A}_{n-1}$ is cast like

$$\mathcal{A}_{n-1}\big(k, p_1, \ldots, p_l, \lambda^2\big) = \mathcal{A}^{\Lambda}_{n-1} + \sum_{i=1}^{n} a_i(k, p_1, \ldots, p_l) \ln^{i-1}\left(-\frac{k^2}{\lambda^2}\right) + \bar{\mathcal{A}}_{n-1}, \tag{3}$$

in which $\bar{\mathcal{A}}_{n-1}$ is finite under integration on k and $\mathcal{A}^{\Lambda}_{n-1}$, the divergent part, represents the subdivergences which in principle are already written in terms of $I^{(i)}_{\log}(\lambda^2)$. The mass scale λ^2 has emerged from a scale relations which characterizes a renormalization scheme in implicit regularization. the coefficients $a_i(k, p_1, \ldots, p_l)$ may contain powers in the external and internal momenta. To justify the assumption of Eq. (3) we proceed with a proof by induction. For $n=2$ (one loop order) it can be easily verified that (3) holds for $\mathcal{A}_1$ [2]. Now we show that this assumption for $(n-1)$th-loop order implies the same structure for the nth-loop order to conclude by induction that the multiloop integrals at any order have the same structure. The relevant contributions come from the second term on the r.h.s. of (3),

$$\int_k^{\Lambda} \frac{k_{\mu_1}\cdots k_{\mu_{r(i)}}}{[(k-p_1)^2-\mu^2]\cdots[(k-p_l)^2-\mu^2]} \ln^{i-1}\left(-\frac{k^2-\mu^2}{\lambda^2}\right), \tag{4}$$

which has superficial degree of divergence $r(i)-2l+4$. Extra factors in the numerator were considered so as to account for the Lorentz structure of the $a_i(k, p_1, \ldots, p_l)$'s. A fictitious mass μ^2 was introduced in the propagators and the limit $\mu^2 \to 0$ will be taken in the end. A fictitious mass may always be introduced if the integral is infrared safe. This is necessary because although the integral is infrared safe, the expansion of the integrand, as we explain below, breaks into infrared-divergent pieces. When a genuine infrared divergence appears, this procedure can be problematic in non-Abelian theories. For such cases a new procedure within IR defining basic infrared-divergent integrals is necessary in order to preserve symmetries [13].

We judiciously apply in the integrand the identity,

$$\frac{1}{(p_r-k)^2-\mu^2} = \frac{1}{(k^2-\mu^2)} - \frac{p_r^2-2p_r\cdot k}{(k^2-\mu^2)[(p_r-k)^2-\mu^2]}, \tag{5}$$