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# Black rings in six dimensions

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ABSTRACT

We propose a general framework for the numerical study of balanced black rings for any spacetime dimensions $d \geqslant 5$. Numerical solutions are constructed in a systematic way for $d = 6$, by solving the Einstein field equations with suitable boundary conditions. These black rings have a regular event horizon with $S^1 \times S^3$ topology, and they approach the Minkowski background asymptotically. We analyze their global and horizon properties.



## 1. Introduction

In 4 dimensions the stationary, asymptotically flat vacuum black holes (BHs) are given by the Kerr family. A spatial section of their event horizon has the topology of a two-sphere $S^2$. The Kerr BHs are uniquely characterized by their mass and angular momentum; thus these two numbers suffice to completely specify a vacuum BH spacetime.

The generalizations of the Kerr BHs to $d \geqslant 5$ dimensions were found by Myers and Perry (MP) [1]. Their global charges are the mass and $N = \lfloor (d-1)/2 \rfloor$ independent angular momenta. Their horizon topology is that of a $(d-2)$-sphere $S^{d-2}$. However, presenting heuristic arguments, Myers and Perry argued that in higher dimensions also black rings (BRs) might exist, and thus black objects with a different horizon topology.

In 2001 Emparan and Reall found such BR solutions in 5 dimensions. These are asymptotically flat and possess a horizon topology $S^2 \times S^1$ [2]. Considering singly rotating BRs, Emparan and Reall showed that, for fixed mass, there are two branches of balanced black rings, a branch of thin black rings and a branch of fat black rings. These two branches merge at a minimal value of the angular momentum $j$, where their horizon area $a_H$ exhibits a cusp. This minimal value of $j$ of the BRs is smaller than the maximal value of $j$ of the MP BHs. Thus, within the range $j^{BR}_{\min} < j < j^{MP}_{\max}$, for given global charges 3 distinct solutions exist. Clearly, uniqueness is violated for these 5-dimensional stationary vacuum solutions.

The discovery of the BRs spurred a lot of interest in BH solutions in higher dimensions. With many more solutions found, such as, in particular, composite objects of BHs and BRs, an intriguing phase diagram of vacuum black objects in 5 dimensions emerged (see e.g. [3,4] for reviews of these aspects). In more than 5 dimensions, however, exact solutions of BRs or composite black objects could not yet be obtained, since no general analytic framework seems to exist for the construction of black objects with nonspherical horizon topology in $d > 5$.

A heuristic way to construct BRs is to bend a Schwarzschild black string and then achieve balance by spinning it along the $S^1$ direction [2]. This may be considered as the underlying picture for approximate techniques, such as the method of matched asymptotic expansion [5,6]. Here the central assumption is that some black objects, in certain ultra-spinning regimes, may be approximated by thin black strings or branes, curved into a given shape and boosted appropriately.

For BRs in $d > 5$ dimensions this approach has led to approximate solutions, valid for configurations with a sufficiently large radius of the ring [5]. However, this approach cannot capture features that are expected to occur at moderate values of the angular momentum, where the radius of the $S^1$ of the ring is no longer large as compared to the radius of the $S^{d-3}$-sphere. Thus, in particular, it cannot deal with the conjectured transition of BRs to BHs with spherical horizon topology [5], i.e., the transition to the branch of pinched black holes emerging from a point of instability of the MP BHs [7,8].

In this work we propose a different approach to the construction of $d > 5$ vacuum BRs, by solving numerically the Einstein equations with suitable boundary conditions. Numerical results are shown for $d = 6$, which allow us to confirm the thin BR part of the phase diagram proposed in [5] and, in addition, to find a branch of fat BRs, which extends towards a horizon topology changing solution.

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## 2. A new coordinate system

A main difficulty in the construction of BR solutions is to find an appropriate coordinate system. The numerical solutions in this work are found for a parametrization of $D$-dimensional flat space (with $D=d-1$)

$$ds_D^2 = V_1\left(dr^2+r^2\,d\theta^2\right)+V_2\,d\Omega_{D-3}^2+V_3\,d\psi^2, \tag{1}$$

where

$$V_1=\frac{1}{U},\qquad V_2=r^2\left(\cos^2\theta-\frac{1}{2}\left(1+\frac{R^2}{r^2}-U\right)\right),$$
$$V_3=r^2\left(\sin^2\theta-\frac{1}{2}\left(1-\frac{R^2}{r^2}-U\right)\right), \tag{2}$$

with $U=\sqrt{1+\frac{R^4}{r^4}-\frac{2R^2}{r^2}\cos 2\theta}$. The coordinate range in (1) is $0\leqslant r<\infty$, $0\leqslant\theta\leqslant\pi/2$, $0\leqslant\psi\leqslant 2\pi$, $d\Omega_{D-3}^2$ is the metric on the unit sphere $S^{D-3}$, and $R>0$ is an arbitrary parameter. The coordinate transformation $\rho=r\sqrt{U}$, $\tan\Theta=(\frac{r^2+\rho^2+R^2}{r^2+\rho^2-R^2})\tan\theta$, leads to a more usual parametrization of the $D>3$ flat space, $ds^2=d\rho^2+\rho^2(d\Theta^2+\cos^2\Theta\,d\Omega_{D-3}^2+\sin^2\Theta\,d\psi^2)$.

It is now manifest that for $0<r<R$, a surface of constant $r$ has ring-like topology $S^{D-2}\times S^1$, where the $S^1$ is parametrized by $\psi$. The BRs will have their event horizon at a constant value of $r<R$, and so they will inherit this topology.[1]

## 3. The ansatz and general relations

The metric for the $d\geqslant 5$ BR geometry preserves most of the basic structure of (1), containing, however, additional terms that encode the gravity effects,

$$ds^2=f_1(r,\theta)\left(dr^2+r^2\,d\theta^2\right)+f_2(r,\theta)\,d\Omega_{d-4}^2 + f_3(r,\theta)\left(d\psi-w(r,\theta)\,dt\right)^2-f_0(r,\theta)\,dt^2. \tag{3}$$

Here the range of the radial coordinate is $r_H\leqslant r<\infty$, and $r=r_H$ corresponds to the event horizon. Thus the domain of integration has a rectangular shape, and is well suited for numerical calculations.

The equations for $(f_i,w)$ are found by using a suitable combination of the Einstein equations, $E_t^t=0$, $E_r^r+E_\theta^\theta=0$, $E_\psi^\psi=0$, $E_\Omega^\Omega=0$ and $E_\psi^t=0$ (with $E_\mu^\nu$ the Einstein tensor), the remaining Einstein equations $E_\theta^r=0$, $E_r^r-E_\theta^\theta=0$ yielding two constraints. The boundary conditions satisfied at $r=r_H$ by the metric functions are $f_0(r_H)=0$, $2f_1+r_H\partial_r f_1=\partial_r f_2=\partial_r f_3=0$, $w=\Omega_H$. As $r\to\infty$, the Minkowski spacetime background is recovered, with $f_0=f_1=1$, $f_2=r^2\cos^2\theta$, $f_3=r^2\sin^2\theta$, $w=0$. At $\theta=\pi/2$, we impose $\partial_\theta f_0=\partial_\theta f_1=f_2=\partial_\theta f_3=\partial_\theta w=0$. The boundary conditions at $\theta=0$ are $\partial_\theta f_0=\partial_\theta f_1=\partial_\theta f_2=f_3=\partial_\theta w=0$, except for the interval $r_H<r\leqslant R$, where we impose instead $f_2=\partial_\theta f_3=0$ on the functions $f_2$, $f_3$.

The metric of a spatial cross-section of the horizon is

$$d\sigma^2=f_1(r_H,\theta)r_H^2\,d\theta^2+f_2(r_H,\theta)\,d\Omega_{d-4}^2+f_3(r_H,\theta)\,d\psi^2. \tag{4}$$

From the above boundary conditions it is clear that the topology of the horizon is $S^{d-3}\times S^1$ (the $S^{d-3}$ is not a round sphere), since $f_3$ is nonzero for any $r\leqslant R$, while $f_2$ vanishes at both $\theta=0$ and $\theta=\pi/2$ (which will correspond to the poles of the $S^{d-3}$-sphere).

[1] Moreover, for $d=5$, the $(r,\theta)$-coordinates correspond to equipotential surfaces of a scalar field sourced by a ring.

The radii on the horizon of the ring circle, $R_1$, and of the $(d-3)$-sphere, $R_{d-3}$, are unambiguously defined only for very thin rings. To obtain a measure for the deformation of the $S^{d-3}$ sphere, we compare the circumference at the equator, $L_e$ ($\theta=\pi/4$, where the sphere is fattest), with the circumference of $S^{d-3}$ along the poles, $L_p$,

$$L_e=2\pi\sqrt{f_2(r_H,\pi/4)},\qquad L_p=2\int_0^{\pi/2}d\theta\,r_H\sqrt{f_1(r_H,\theta)}, \tag{5}$$

and consider, in particular, their ratio $L_e/L_p$. An estimate of the deformation of $S^1$ is given by the ratio $R_1^{(in)}/R_1^{(out)}$, where $R_1^{(in)}$ and $R_1^{(out)}$ are its radii inside and outside of the ring, respectively,

$$R_1^{(in)}=\sqrt{f_3(r_H,0)},\qquad R_1^{(out)}=\sqrt{f_3(r_H,\pi/2)}. \tag{6}$$

A study of the $d=5$ Emparan–Reall BR written within the ansatz (3) can be found in Ref. [9], including the explicit form of the metric functions. Note, that the $d\geqslant 5$ MP BH with one rotation parameter can also be written in the form (3). For BHs with a spherical horizon topology, the metric functions satisfy the same set of boundary conditions, except for $f_2$ and $f_3$ at $\theta=0$, where $\partial_\theta f_2=f_3=0$ for any $r>r_H$ (see Ref. [9] for a discussion of the $d=5$ case).

For both BRs and MP BHs, the event horizon area $A_H$, Hawking temperature $T_H$ and event horizon velocity $\Omega_H$ of the solutions are given by

$$A_H=2\pi r_H V_{d-4}\int_0^{\pi/2}d\theta\,\sqrt{f_1 f_2^{d-4} f_3}\bigg|_{r=r_H},$$
$$T_H=\frac{1}{2\pi}\lim_{r\to r_H}\frac{1}{(r-r_H)}\sqrt{\frac{f_0}{f_1}},\qquad \Omega_H=w|_{r=r_H}, \tag{7}$$

where $V_{d-4}$ is the area of the unit $S^{d-4}$ sphere.

The mass and angular momentum are read from the large-$r$ asymptotics of the metric functions, $g_{tt}=-f_0=-1+\frac{C_t}{r^{d-3}}+\cdots$, $g_{\psi t}=-f_3 w=\sin^2\theta\frac{C_\psi}{r^{d-3}}+\cdots$, with $(G=1)$

$$M=\frac{(d-2)V_{d-2}}{16\pi}C_t,\qquad J=\frac{V_{d-2}}{8\pi}C_\psi. \tag{8}$$

Also, both the MP and the BR solutions satisfy the Smarr law

$$\frac{d-3}{d-2}M=T_H\frac{A_H}{4}+\Omega_H J. \tag{9}$$

Following [5], we define a scale by fixing the mass and introduce the dimensionless 'reduced' quantities

$$j=c_j^{\frac{1}{d-3}}\frac{J}{M^{\frac{d-2}{d-3}}},\qquad a_H=c_a^{\frac{1}{d-3}}\frac{A_H}{M^{\frac{d-2}{d-3}}},$$
$$w_H=c_w\Omega_H M^{\frac{1}{d-3}},\qquad t_H=c_t T_H M^{\frac{1}{d-3}}, \tag{10}$$

where $c_j=\frac{V_{d-3}}{2^{d+1}}\frac{(d-2)^{d-2}}{(d-3)^{(d-3)/2}}$, $c_a=\frac{V_{d-3}}{2(16\pi)^{d-3}}(d-2)^{d-2}(\frac{d-4}{d-3})^{\frac{d-3}{2}}$, $c_w=\sqrt{d-3}(\frac{(d-2)}{16}V_{d-3})^{-\frac{1}{d-3}}$, $c_t=4\pi\sqrt{\frac{d-3}{d-4}}(\frac{d-2}{2}V_{d-3})^{-\frac{1}{d-3}}$. Then both the BRs and the MP BHs are conveniently parametrized by a single dimensionless parameter which we choose to be $j$.

## 4. The numerical scheme

We employ a numerical algorithm developed in [9,10] which uses a Newton–Raphson method to solve for $(f_i,w)$, whilst