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The Boltzmann equation from quantum field theory

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ABSTRACT

We show from first principles the emergence of classical Boltzmann equations from relativistic nonequilibrium quantum field theory as described by the Kadanoff–Baym equations. Our method applies to a generic quantum field, coupled to a collection of background fields and sources, in a homogeneous and isotropic spacetime. The analysis is based on analytical solutions to the full Kadanoff–Baym equations, using the WKB approximation. This is in contrast to previous derivations of kinetic equations that rely on similar physical assumptions, but obtain approximate equations of motion from a gradient expansion in momentum space. We show that the system follows a generalized Boltzmann equation whenever the WKB approximation holds. The generalized Boltzmann equation, which includes off-shell transport, is valid far from equilibrium and in a time dependent background, such as the expanding universe.

1. Introduction

Nonequilibrium phenomena play a crucial role in many areas of physics, including the early history of the universe, heavy ion collisions, condensed matter physics and quantum information. In the era of precision cosmology and with the arrival of the LHC and RHIC experiments, in particular the first two applications, which require a relativistic description, have gained considerable interest. Transport in nonequilibrium situations can often in very good approximation be described by Boltzmann equations (BEs). These assume that the system can be characterized by a number of distribution functions for classical particles, which propagate freely between isolated interactions and carry no memory of their history. However, the definition of asymptotic states, on which the single particle description is based, is ambiguous in a dense plasma. What is more, the standard BEs by construction cannot describe memory and off-shell effects or quantum coherence. Usually these effects are treated by effective kinetic equations of the Boltzmann type [1–33], i.e. by a set of first order differential equations for generalized distribution functions that are local in time. As the above issues are conceptual, their range of validity and possible corrections cannot be determined within a framework of BEs and require a derivation from first principles.

The full equations of motion of nonequilibrium quantum field theory, on which first-principle derivations of the BEs are usually based, are known as Kadanoff–Baym equations (KBEs) [3].[1] These equations, being coupled second order integro-differential equations, are considerably more complicated than BEs. Most approaches to establish a connection between out-of-equilibrium quantum fields and kinetic equations make a number of approximations on the KBEs *before* they are solved (e.g. Refs. [3,5,6,9,11–15,17–19,23–25,27]). Starting point is usually a gradient expansion, performed in Wigner-space, which provides a consistent approximation scheme when a separation of scales is realized in the system. Common additional simplifications include close-to-equilibrium assumption for all fields, the quasiparticle approximation and the Kadanoff–Baym ansatz for correlation functions. However, the Wigner space method as such does not rely on these additional assumptions if the gradient expansion is performed consistently, which may require resummations [28].

In this Letter, we show how the *full* KBEs can be solved by using the Wentzel–Kramers–Brillouin (WKB) [34] method (for earlier uses of the WKB method in a similar context see e.g. Refs. [13,15]). This approach avoids the Fourier transformation to Wigner space in relative time and uses what is sometimes called the *two time formalism*. It is valid far from equilibrium and does not rely on

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[1] The KBEs are equations of motion for correlation functions. Alternatively one can use equations of motion for the fields themselves as starting point, cf. [33] and references therein for a detailed comparison.

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an on-shell approximation or any other a priori assumption about the form of the correlation functions, such as the Kadanoff–Baym ansatz. We illustrate our method for a real scalar field, coupled to other fields, in a spatially homogeneous and isotropic background. This choice is for transparency only; the derivation does not rely on assumptions about the spin and interactions of the field or background. Though technically more difficult, the generalization to fermions with gauge interactions is straightforward.

2. Nonequilibrium quantum field theory

We consider the dynamics of a real scalar field ϕ that is described by relativistic quantum field theory. The field ϕ weakly couples to a background, possibly containing many degrees of freedom whose dynamics is in principle known and that we refer to as χ_i. The Lagrangian reads

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m(t)^2\phi^2 - \phi\mathcal{O}[\chi_i,t] + \mathcal{L}_{\chi_i}, \tag{1}$$

where $\mathcal{O}[\chi_i,t]$ denotes the sum of generic combinations of fields χ_i with coefficients that may depend on time explicitly.[2] $\mathcal{L}_{\chi_i}$ determines the dynamics of χ_i (we use $\hbar = c = 1$). We allow a time-dependent mass $m(t)$ to account for Hubble expansion when interpreting t as conformal time, the time-dependence of other operators is contained in $\mathcal{O}[\chi_i,t]$ and $\mathcal{L}_{\chi_i}$.

In quantum physics, any thermodynamic system can be characterized by a density matrix ϱ. Knowledge of the density matrix allows to compute expectation values for all observables at all times. The same information is contained in the set of all n-point functions $\langle\phi(x_1)\cdots\phi(x_n)\rangle$ etc. of the fields (with $\langle\cdots\rangle \equiv \mathrm{tr}[\varrho\cdots]$). However, most quantities of practical interest for which one formulates a Boltzmann equation can be expressed in terms of one- and two-point functions; this includes the energy–momentum tensor and charge densities. It is, therefore, usually sufficient to track the time evolution of these.

An out-of-equilibrium quantum field has two independent connected two-point functions. In case of ϕ they are conveniently chosen as

$$\begin{aligned}\Delta^-(x_1,x_2) &\equiv i\big\langle\big[\phi(x_1),\phi(x_2)\big]\big\rangle,\\ \Delta^+(x_1,x_2) &\equiv \frac{1}{2}\big\langle\big\{\phi(x_1),\phi(x_2)\big\}\big\rangle,\end{aligned} \tag{2}$$

with the obvious symmetry relations $\Delta^\pm(x_2,x_1) = \pm\Delta^\pm(x_1,x_2)$. Here $[\,,]$ and $\{\,,\}$ are commutator and anti-commutator, respectively. $\Delta^-(x_1,x_2)$ is known as *spectral function* and basically encodes information about the spectrum of resonances in the thermodynamic description, which may differ from the spectrum in vacuum. The *statistical propagator* $\Delta^+(x_1,x_2)$ carries information about the occupation numbers of different modes. We will in the following derive the quantum field theory analogue to the classical particle distribution function from the statistical propagator. We have in mind applications in cosmology and restrict the analysis to spatially homogeneous and isotropic systems. Then, the correlation functions only depend on relative spatial coordinates $\mathbf{x}_1 - \mathbf{x}_2$ etc., and it is convenient to perform a spatial Fourier transform in these coordinates, yielding functions like $\Delta^\pm_{\mathbf{q}}(t_1,t_2) \equiv \int d^3(\mathbf{x}_1-\mathbf{x}_2)\, e^{-i\mathbf{q}(\mathbf{x}_1-\mathbf{x}_2)}\Delta^\pm(x_1,x_2)$.

[2] Our approach does not rely on the way ϕ couples to the bath as long as conditions (1)–(3) specified in Section 3 are fulfilled. In (1) we chose a coupling that is linear in ϕ to obtain the simple explicit expressions (5) for the self-energies and to justify the time translation invariance of $\Pi^\pm$ in Eqs. (24)–(26). All other formulae and considerations remain valid for an arbitrary coupling between ϕ and other fields.

In a general out-of-equilibrium system the two-point functions $\Delta^\pm_{\mathbf{q}}(t_1,t_2)$ have to be found as solutions to the KBEs

$$\big(\partial_{t_1}^2 + \omega_{\mathbf{q}}(t_1)^2\big)\Delta^-_{\mathbf{q}}(t_1,t_2) = -\int_{t_2}^{t_1} dt'\,\Pi^-_{\mathbf{q}}\big(t_1,t'\big)\Delta^-_{\mathbf{q}}\big(t',t_2\big), \tag{3}$$

$$\begin{aligned}&\big(\partial_{t_1}^2 + \omega_{\mathbf{q}}(t_1)^2\big)\Delta^+_{\mathbf{q}}(t_1,t_2)\\ &\quad= -\int_{t_i}^{t_1} dt'\,\Pi^-_{\mathbf{q}}\big(t_1,t'\big)\Delta^+_{\mathbf{q}}\big(t',t_2\big) + \int_{t_i}^{t_2} dt'\,\Pi^+_{\mathbf{q}}\big(t_1,t'\big)\Delta^-_{\mathbf{q}}\big(t',t_2\big),\end{aligned} \tag{4}$$

where $\omega_{\mathbf{q}}(t)^2 \equiv m(t)^2 + \mathbf{q}^2$ (note that in equilibrium $\Delta^\pm_{\mathbf{q}}$ would only depend on $t_1 - t_2$), and t_i denotes the initial time of the system. The KBEs can be derived within the Schwinger–Keldysh formalism, see e.g. [6,35–38]. The first term on the RHS of (4) is associated with non-Markovian (memory) effects while the second is often referred to as noise term. The boundary conditions for $\Delta^-_{\mathbf{q}}$ are fixed by microcausality and canonical quantization for a real scalar field, $\Delta^-_{\mathbf{q}}|_{t_1=t_2} = 0$, $\partial_{t_1}\Delta^-_{\mathbf{q}}|_{t_1=t_2} = -\partial_{t_2}\Delta^-_{\mathbf{q}}|_{t_1=t_2} = 1$ and $\partial_{t_1}\partial_{t_2}\Delta^-_{\mathbf{q}}|_{t_1=t_2} = 0$. The boundary conditions for $\Delta^+_{\mathbf{q}}|_{t_1=t_2=t_i}$ are determined by the physical initial conditions of the system at time t_i. For simplicity we assumed Gaussian initial correlations for ϕ, more general initial conditions are e.g. discussed in [22]. Below, we will drop momentum indices $\mathbf{q}$ when possible.

The quantities $\Pi^\pm$ appearing in (3) and (4) are the self-energies of ϕ; in analogy to (2) they are at leading order in $\mathcal{O}[\chi_i]$ given by

$$\begin{aligned}\Pi^-(x_1,x_2) &= \big\langle\big[\mathcal{O}\big[\chi_i(x_1),t_t\big],\mathcal{O}\big[\chi_i(x_2),t_2\big]\big]\big\rangle,\\ \Pi^+(x_1,x_2) &= -\frac{i}{2}\big\langle\big\{\mathcal{O}\big[\chi_i(x_1),t_1\big],\mathcal{O}\big[\chi_i(x_2),t_2\big]\big\}\big\rangle,\end{aligned} \tag{5}$$

and contain information about the interaction between ϕ and the background fields χ_i. They can be calculated in terms of the two-point functions of χ_i within the 2PI formalism (see e.g. Refs. [35, 36] for details).

3. Towards Boltzmann equations

We will discuss the emergence of a description of ϕ in terms of effective kinetic equations by using analytical solutions of the full KBEs that are found with the WKB method. To this end, we make the following assumptions (we also send $t_i \to -\infty$, effects of finite t_i are discussed below):

(1) The self-energies $\Pi^\pm(t_1,t_2)$ are damped with respect to the relative time $|t_1 - t_2|$, approaching zero for $|t_1 - t_2| \gtrsim \tau_{\mathrm{int}}$, where we introduced the *interaction time* τ_{int}. Here, τ_{int} can be considered as definition for the duration of e.g. scattering events. Then evaluating one-sided Fourier transforms of the self-energies with respect to relative time,

$$\widetilde{\Pi}^\pm(t,\omega) \equiv \int_0^\infty dz\, e^{i\omega z}\Pi^\pm(t,t-z), \tag{6}$$

practically does not require knowledge of the system in the distant past $z \gg \tau_{\mathrm{int}}$. In equilibrium the minus-component would correspond to the common retarded self-energy, $\widetilde{\Pi}^-(t,\omega) = \Pi^R(\omega)$.

(2) We assume that for fixed time t the pole structure of $(\omega^2 - \omega^2_{\mathbf{q}}(t) - \widetilde{\Pi}^-(t,\omega))^{-1}$ is dominated by the root $\omega = \widehat{\Omega}_t \equiv \Omega_t - \frac{i}{2}\Gamma_t$, with

$$\Omega_t \equiv \sqrt{\omega^2_{\mathbf{q}}(t) + \mathrm{Re}\,\widetilde{\Pi}^-(t,\widehat{\Omega}_t)}, \qquad \Gamma_t \equiv -\frac{\mathrm{Im}\,\widetilde{\Pi}^-(t,\widehat{\Omega}_t)}{\Omega_t}. \tag{7}$$