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The neutrino mixing matrix could (almost) be diagonal with entries ±1

Yoni BenTov [a,*], A. Zee [a,b]

[a] Department of Physics, University of California, Santa Barbara, CA 93106, United States
[b] Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, United States

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ABSTRACT

It is consistent with the measurement of $\theta_{13} \sim 0.15$ by Daya Bay to suppose that, in addition to being unitary, the neutrino mixing matrix is also almost Hermitian, and thereby only a small perturbation from diag$(+1, -1, -1)$ in a suitable basis. We suggest this possibility simply as an easily falsifiable ansatz, as well as to offer a potentially useful means of organizing the experimental data. We explore the phenomenological implications of this ansatz and parametrize one type of deviation from the leading order relation $|V_{e3}| \approx |V_{\tau 1}|$. We also emphasize the group-invariant angle between orthogonal matrices as a means of comparing to data. The discussion is purely phenomenological, without any attempt to derive the condition $V^\dagger \approx V$ from a fundamental theory.

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1. A phenomenological ansatz

The neutrino mixing matrix V is defined by $\nu_\alpha = V_{\alpha i}\nu_i$, where $\alpha = e, \mu, \tau$ denotes the charged lepton mass basis ("flavor basis"), and $i = 1, 2, 3$ denotes the neutrino mass basis. The relevant part of the Lagrangian written in the flavor basis reads

$$\mathcal{L} = -\sum_{\alpha=e,\mu,\tau} m_\alpha e_\alpha \bar{e}_\alpha - \frac{1}{2}\nu_\alpha (M_\nu)_{\alpha\beta}\nu_\beta + \text{h.c.} \tag{1.1}$$

In this basis, the neutrino mass matrix is $M_\nu = V^* D_\nu V^\dagger$, where $D_\nu \equiv \text{diag}(m_1, m_2, m_3)$ with the m_i real and positive.

Assuming that the three light neutrinos of the Standard Model are Majorana, the magnitudes of the entries of V are constrained by data to be

$$|V_{\text{exp}}| \approx \begin{pmatrix} 0.78\text{–}0.84 & 0.52\text{–}0.61 & 0.13\text{–}0.17 \\ 0.40\text{–}0.58 & 0.39\text{–}0.65 & 0.57\text{–}0.80 \\ 0.19\text{–}0.43 & 0.53\text{–}0.74 & 0.59\text{–}0.81 \end{pmatrix}. \tag{1.2}$$

To obtain Eq. (1.2) we have used $0.550 \leqslant \theta_{12} \leqslant 0.658$ and $0.620 \leqslant \theta_{23} \leqslant 0.934$ from the work of Gonzalez-Garcia, Maltoni, and Salvado [1], and $0.135 \leqslant \theta_{13} \leqslant 0.171$ from the recent results of Daya Bay [2]. The ranges in Eq. (1.2) are correlated in such a way as to preserve the unitarity condition $V^\dagger V = I$.

In an effort to obtain a theoretical understanding of the mixing matrix, one might suppose that the numerical values in Eq. (1.2) arise as a small perturbation from a "simple" ansatz. As a straw man argument for what such an ansatz might be, consider an older global best fit given by [3]

$$|V_{\text{exp, old}}| \approx \begin{pmatrix} 0.77\text{–}0.86 & 0.50\text{–}0.63 & 0.00\text{–}0.22 \\ 0.22\text{–}0.56 & 0.44\text{–}0.73 & 0.57\text{–}0.80 \\ 0.21\text{–}0.55 & 0.40\text{–}0.71 & 0.59\text{–}0.82 \end{pmatrix}. \tag{1.3}$$

Simply by looking at the ranges in Eq. (1.3), we observe that it was once numerically consistent to suppose that V is Hermitian. Without any theoretical motivation, we now suppose that the true mixing matrix satisfies the leading order relation $V^\dagger \approx V$, and then we study small deviations required to fit the new data. We propose this rather ad hoc constraint in the spirit of trying to make sense of the data by reducing the number of free parameters in the neutrino sector [4]. This exercise is intended partially to illustrate that there are still many possibilities for what the true mixing matrix might be.

2. Real symmetric mixing matrix

As a warmup, we first consider the case for which V is real. Then V is orthogonal, meaning $V^T V = I$, and our Hermiticity ansatz amounts to imposing the condition $V^T = V$. The old experimental bounds adjusted for compatibility with this ansatz are

$$|V^{\text{ansatz}}_{\text{exp, old}}| \approx \begin{pmatrix} 0.77\text{–}0.86 & 0.50\text{–}0.56 & 0.21\text{–}0.22 \\ 0.50\text{–}0.56 & 0.44\text{–}0.73 & 0.57\text{–}0.71 \\ 0.21\text{–}0.22 & 0.57\text{–}0.71 & 0.59\text{–}0.82 \end{pmatrix}. \tag{2.1}$$

Again, the ranges are correlated, as required by $V^T V = I$ [see Eq. (2.4)]. Immediately we see that the Daya Bay observation of

* Corresponding author.
E-mail address: bentov@physics.ucsb.edu (Y. BenTov).

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$|V_{e3}| < 0.21$ rules out this ansatz as an exact prediction,[1] but otherwise it is still consistent with Eq. (1.2).

If V is real symmetric, then it can be diagonalized by an orthogonal transformation: $V = XdX^T$, where d is diagonal and X is orthogonal. Then $V^TV = I$ implies $V^2 = Xd^2X^T = I$, so that $d^2 = I$. Thus our ansatz amounts to proposing that, in a particular basis, the neutrino mixing matrix is diagonal with entries equal to ± 1.

We now have a choice as to how to arrange the minus signs in d. Two of the nonzero entries in d must have the same sign, while the third must have a sign opposite to that of the first two.[2] In other words, we get to choose which 2-dimensional subspace of d is proportional to the identity matrix. This choice is arbitrary[3]; to fix the discussion, we choose

$$d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{2.2}$$

so that d equals minus the identity matrix in the $(2,3)$-plane.

Any rotation matrix in 3 dimensions can be written as a product of independent rotations about each of 3 mutually orthogonal axes. That is, given the rotation matrices

$$X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_1 & -S_1 \\ 0 & S_1 & C_1 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} C_2 & 0 & S_2 \\ 0 & 1 & 0 \\ -S_2 & 0 & C_2 \end{pmatrix},$$
$$X_3 = \begin{pmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.3}$$

where $C_I \equiv \cos\varphi_I$ and $S_I \equiv \sin\varphi_I$, we can write X as a product of the three X_I in any order.[4] Since d is proportional to the identity matrix in the $(2,3)$-plane, it is unchanged by a rotation about the first axis: $X_1 d X_1^T = d$. Thus one of the parameters in $V = XdX^T$ drops out, leaving us with a two-parameter ansatz for the mixing matrix. Choosing the ordering $X = X_3X_2X_1$ implies

$$V = \begin{pmatrix} C_2^2\cos(2\varphi_3) - S_2^2 & -C_2^2\sin(2\varphi_3) & -C_3\sin(2\varphi_2) \\ \times & -C_2^2\cos(2\varphi_3) - S_2^2 & S_3\sin(2\varphi_2) \\ \times & \times & -\cos(2\varphi_2) \end{pmatrix} \tag{2.4}$$

where we have displayed only the upper triangle of V since by construction it is symmetric. The values for the angles consistent with Eq. (1.3) turn out to be $0.32 \leqslant \varphi_2 \leqslant 0.42$ and $1.20 \leqslant \varphi_3 \leqslant 1.27$, where the angles are expressed in radians, as shown in Fig. 1.

As an arbitrarily chosen "typical" example of Hermitian mixing, the values $(\varphi_2, \varphi_3) = (0.35, 1.23)$ imply a mixing matrix

$$V_{0.35,1.23} \approx \begin{pmatrix} -0.80 & 0.56 & 0.22 \\ 0.56 & 0.57 & 0.61 \\ 0.22 & 0.61 & -0.76 \end{pmatrix} \tag{2.5}$$

[1] Much of this work was completed before the measurement of nonzero reactor angle, $\theta_{13} \sim 0.15$. The fact that the reactor angle is relatively large, meaning closer to ~ 0.2 than to zero, is what keeps our analysis relevant.

[2] The solution $d = I$ trivially satisfies $d^2 = I$. This would result in $V = XdX^T = XX^T = I$, which is of course experimentally unacceptable.

[3] For example, let $X = X'P$, where X' is orthogonal and P is a permutation matrix. Then $V \equiv XdX^T = X'd'(X')^T$ is of the same form as before, but with a new diagonal matrix $d' = PdP^T$ with the two minus signs permuted. Of course, we are also free to multiply V and hence d by an overall sign.

[4] At this point we should emphasize that φ_I are not the three PMNS angles that parametrize the mixing matrix $V = XdX^T$. That is why we have chosen to denote their sines and cosines by capital letters, in contrast to the notation in Section 3 for the usual PMNS angles.

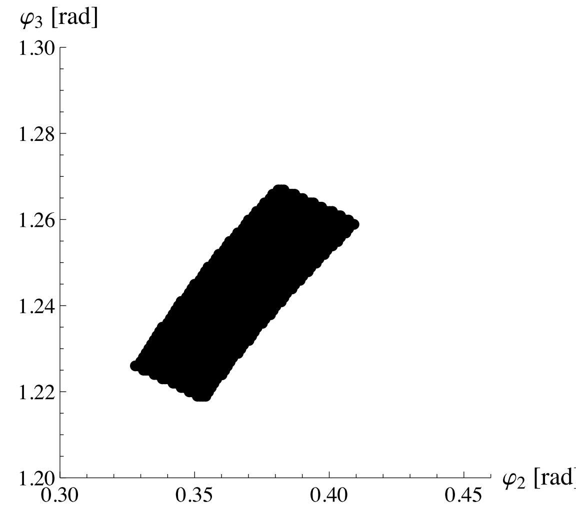

Fig. 1. The values of φ_2 and φ_3 in the quadrant $(\varphi_2, \varphi_3) \in ([0, \frac{\pi}{2}], [0, \frac{\pi}{2}])$ consistent with an older set of oscillation data, Eq. (1.3). This may be a useful starting point about which to perturb in order to fit the new data, Eq. (1.2).

where we have rearranged the minus signs into a standard form.[5] Compare this with the often-studied "tribimaximal mixing" ansatz [5,6]

$$V_{\mathrm{TB}} \equiv \begin{pmatrix} \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \approx \begin{pmatrix} -0.82 & 0.58 & 0 \\ 0.41 & 0.58 & 0.71 \\ 0.41 & 0.58 & -0.71 \end{pmatrix}. \tag{2.6}$$

These two matrices appear qualitatively "very different," given that one has $V_{e3} \approx 0.22$ while the other has $V_{e3} = 0$. To make this notion more precise, define[6] the $SO(3)$-invariant angle Θ between two special orthogonal matrices V and V':

$$\Theta(V, V') \equiv \cos^{-1}\left(\frac{1}{3}\operatorname{tr}(V^TV')\right). \tag{2.7}$$

The matrices $V_{0.35,1.23}$ and V_{TB} are separated by an angle

$$\Theta(V_{0.35,1.25}, V_{\mathrm{TB}}) \approx 0.20 \sim 11^\circ \tag{2.8}$$

in $SO(3)$. As a related example, one might compare to another ansatz with the same atmospheric and reactor angles as tribimaximal mixing ($\theta_{23} = \frac{\pi}{4}$ and $\theta_{13} = 0$, respectively), but with the solar angle related to[7] the golden ratio: $\theta_{12} = \tan^{-1}(1/\varphi)$, with $\varphi = \frac{1}{2}(1 + \sqrt{5})$ [7–11]. The PMNS matrix for this ansatz is

$$V_{\mathrm{golden}} \approx \begin{pmatrix} -0.85 & 0.53 & 0 \\ 0.37 & 0.60 & 0.71 \\ 0.37 & 0.60 & -0.71 \end{pmatrix}. \tag{2.9}$$

This is separated from the matrix of Eq. (2.5) by an angle $\Theta(V_{0.35,1.25}, V_{\mathrm{golden}}) \approx 0.20 \sim 11^\circ$, approximately the same as for V_{TB}. More generally, we see that the entire family of "$\mu\tau$-symmetric" mixing matrices is approximately separated from the entire family of Hermitian mixing matrices by $\sim 11^\circ$ in $SO(3)$.

[5] So as not to interrupt the logical flow we will postpone discussion of rephasing V until Section 3.

[6] Another measure [12] of the $SO(3)$-invariant distance between matrices is $\mathcal{D}(V, V') = \frac{1}{3}\operatorname{tr}(I - V^TV')$. Here we choose the angular distance because it provides an intuitive notion of "large" versus "small" in terms of an angle Θ ranging from 0 to $\pi/2$.

[7] The references discuss various different proposals for relating the solar angle to the golden ratio. We arbitrarily choose the particular implementation of Eq. (2.9) to be concrete.