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Operator equations and Moyal products–metrics in quasi-Hermitian quantum mechanics

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Abstract

The Moyal product is used to cast the equation for the metric of a non-Hermitian Hamiltonian in the form of a differential equation. For Hamiltonians of the form $p^2 + V(ix)$ with V polynomial this is an exact equation. Solving this equation in perturbation theory recovers known results. Explicit criteria for the hermiticity and positive definiteness of the metric are formulated on the functional level.

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1. Introduction

The recent interest in PT-symmetric quantum mechanics [1,2] has revitalized the old question [3–5] of the existence of a metric and associated inner product for which a standard quantum mechanical interpretation is possible, even though the Hamiltonian may be non-Hermitian with respect to the given inner product. Here we address this issue with technology borrowed from non-commutative quantum mechanics. The advantage of this approach is that the operator equation that must be solved can often be cast in the form of a differential equation without making any approximations. Generically this equation may be of infinite order, but in many cases of physical interest it turns out to be finite. This equation contains all the information required to construct the metric operator exactly. In addition criteria can be formulated to test the hermiticity and positive definiteness of the metric directly on the level of this equation, leading to considerable simplification. On this level the non-uniqueness of the metric is reflected in the choice of boundary conditions. On the other hand it is known [3] that the metric is uniquely determined (up to an irrelevant normalization factor) once a complete set of irreducible observables has been specified which is Hermitian with respect to the inner product associated with the metric. This suggests an interplay between boundary conditions in phase space and the choice of physical observables.

2. A Moyal product primer

2.1. Finite-dimensional Hilbert space

Although the Moyal product is a well established tool [6], recently revived in the context of non-commutative systems (see e.g. [7]), we review the construction briefly in order to adapt it to our specific application. We start by considering the construction

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of an irreducible unitary representation of the Heisenberg–Weyl algebra

$$gh = e^{i\phi} hg, \qquad g^\dagger = g^{-1}, \qquad h^\dagger = h^{-1} \tag{1}$$

on a finite-dimensional Hilbert space with dimension N. Clearly, such a representation can only exist for non-trivial ϕ if $\operatorname{tr} g = 0$. The implication of this is clear when we compute the trace explicitly. Since g is unitary,

$$g|\alpha\rangle = e^{i\alpha}|\alpha\rangle, \quad \alpha \in R, \qquad \langle\alpha'|\alpha\rangle = \delta_{\alpha',\alpha}. \tag{2}$$

From (1) it follows that

$$gh|\alpha\rangle = e^{i(\alpha+\phi)} h|\alpha\rangle, \tag{3}$$

so that h ladders between the eigenvalues of g. We conclude that the eigenvalues and eigenstates of g are of the form

$$|\alpha_0 + n\phi\rangle = h^n|\alpha_0\rangle \equiv |n\rangle, \qquad g|\alpha_0 + n\phi\rangle = e^{i(\alpha_0+n\phi)}|\alpha_0 + n\phi\rangle, \tag{4}$$

with α_0 an arbitrary constant (set to zero without loss of generality). If the representation is to be irreducible, all N orthogonal eigenstates of g can be reached through such a laddering process.

It is now simple to compute $\operatorname{tr} g = 0$. The result is

$$\operatorname{tr} g = \sum_{n=0}^{N} e^{in\phi} = \frac{1 - e^{iN\phi}}{1 - e^{i\phi}}, \tag{5}$$

which vanishes only when $\phi = 2m\pi/N$ with m integer. This limits the allowed values of ϕ. Note from (4) that $g^N|n\rangle = |n\rangle$, $h^N|n\rangle = |n\rangle$, $\forall n$, implying $g^N = h^N = 1$. The operators g^n and h^m are therefore only independent when $n, m < N$. Motivated by this, we choose $\phi = 2\pi/N$ as with this choice the operators $U(n,m) \equiv g^n h^m$, with $n = 0, 1, \ldots, N-1$ and $m = 0, 1, \ldots, N-1$, form a basis in the space of operators (matrices) on the Hilbert space. To show the linear independence and completeness of this basis we introduce the standard inner product on the space of operators

$$(A, B) = \operatorname{tr} A^\dagger B. \tag{6}$$

It immediately follows from (4) that $(U(n',m'), U(n,m)) = N\delta_{n',n}\delta m', m$, implying that these operators are linearly independent. As there are N^2 such complex linearly independent operators, it follows on simple dimensional grounds that they provide a basis.

The conditions above are necessary for the existence of a representation of (1), but we have not yet demonstrated that such a representation actually exists. This follows from explicit construction. It is easy to verify that the following matrices satisfy all the conditions above [8]

$$g_{n,m} = e^{\frac{2\pi i(n-1)}{N}}\delta_{n,m}, \qquad h_{n,m} = \delta_{n,m-1} + \delta_{n,N}\delta_{m,1}. \tag{7}$$

As the operators $U(n,m) \equiv g^n h^m$ form a basis, any operator A can be expanded in the form

$$A = \sum_{n,m=0}^{N-1} a_{n,m} g^n h^m, \quad a_{n,m} = \big(U(n,m), A\big)/N. \tag{8}$$

Consider the multiplication of two operators A and B

$$AB = \sum_{n,m=0}^{N-1} \sum_{n',m'=0}^{N-1} a_{n,m} b_{n',m'} e^{-imn'\phi} g^{n+n'} h^{m+m'}. \tag{9}$$

Apart from the phase $e^{-imn'\phi}$ this looks like the multiplication of two sums in which g and h are treated as ordinary complex numbers. One may therefore take the point of view that g and h are to be treated as complex numbers, but then the product rule must be modified to ensure equivalence with (9). Making the following substitutions

$$g \to e^{i\alpha}, \qquad h \to e^{i\beta}, \quad \alpha,\ \beta \in [0, 2\pi) \tag{10}$$

in the expansion (8), turns A into a function $A(\alpha,\beta)$, uniquely determined by the operator A

$$A = \sum_{n,m=0}^{N-1} a_{n,m} e^{in\alpha} e^{im\beta}. \tag{11}$$

To establish an isomorphism with the product (9) we define the Moyal product of functions $A(\alpha,\beta)$ and $B(\alpha,\beta)$ [6,7]

$$A(\alpha,\beta) * B(\alpha,\beta) = A(\alpha,\beta) e^{i\phi \overleftarrow{\partial}_\beta \overrightarrow{\partial}_\alpha} B(\alpha,\beta), \tag{12}$$