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Electron–positron pairs production in a macroscopic charged core

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ABSTRACT

Classical and semi-classical energy states of relativistic electrons bounded by a massive and charged core with the charge-mass radio Q/M and macroscopic radius R_c are discussed. We show that the energies of semi-classical (bound) states can be much smaller than the negative electron mass-energy ($-mc^2$), and energy-level crossing to negative energy continuum occurs. Electron–positron pair production takes place by quantum tunneling, if these bound states are not occupied. Electrons fill into these bound states and positrons go to infinity. We explicitly calculate the rate of pair-production, and compare it with the rates of electron–positron production by the Sauter–Euler–Heisenberg–Schwinger in a constant electric field. In addition, the pair-production rate for the electro-gravitational balance ratio $Q/M = 10^{-19}$ is much larger than the pair-production rate due to the Hawking processes.

1. Introduction

As reviewed in the recent report [1], very soon after the Dirac equation for a relativistic electron was discovered [2,3], Gordon [4] (for all $Z < 137$) and Darwin [5] (for $Z = 1$) found its solution in the point-like Coulomb potential $V(r) = -Z\alpha/r$, they obtained the well-known Sommerfeld's formula [6] for energy spectrum,

$$\mathcal{E}(n,j) = mc^2\left[1+\left(\frac{Z\alpha}{n-|K|+(K^2-Z^2\alpha^2)^{1/2}}\right)^2\right]^{-1/2}, \tag{1}$$

where the fine-structure constant $\alpha = e^2/\hbar c$, the principle quantum number $n = 1, 2, 3, \dots$ and

$$K = \begin{cases} -(j+1/2) = -(l+1), & \text{if } j = l+\frac{1}{2},\ l \geqslant 0, \\ (j+1/2) = l, & \text{if } j = l-\frac{1}{2},\ l \geqslant 1, \end{cases} \tag{2}$$

$l = 0, 1, 2, \dots$ is the orbital angular momentum corresponding to the upper component of Dirac bi-spinor, j is the total angular momentum. The integer values n and j label bound states whose energies are $\mathcal{E}(n,j) \in (0, mc^2)$. For the example, in the case of the lowest energy states, one has

$$\mathcal{E}(1S_{\frac{1}{2}}) = mc^2\sqrt{1-(Z\alpha)^2}, \tag{3}$$

$$\mathcal{E}(2S_{\frac{1}{2}}) = \mathcal{E}(2P_{\frac{1}{2}}) = mc^2\sqrt{\frac{1+\sqrt{1-(Z\alpha)^2}}{2}}, \tag{4}$$

$$\mathcal{E}(2P_{\frac{3}{2}}) = mc^2\sqrt{1-\frac{1}{4}(Z\alpha)^2}. \tag{5}$$

For all states of the discrete spectrum, the binding energy $mc^2 - \mathcal{E}(n,j)$ increases as the nuclear charge Z increases. No regular solution with $n = 1$, $l = 0$, $j = 1/2$ and $K = -1$ (the $1S_{1/2}$ ground state) is found for $Z > 137$, this was first noticed by Gordon in his pioneer paper [4]. This is the problem so-called "$Z = 137$ catastrophe".

The problem was solved [7–14] by considering the fact that the nucleus is not point-like and has an extended charge distribution, and the potential $V(r)$ is not divergent when $r \to 0$. The $Z = 137$ catastrophe disappears and the energy-levels $\mathcal{E}(n,j)$ of the bound states $1S$, $2P$ and $2S$, ... smoothly continue to drop toward the negative energy continuum ($E_- < -mc^2$), as Z increases to values larger than 137. The critical values Z_{cr} for $\mathcal{E}(n,j) = -mc^2$ were found [9,11–14,17–19]: $Z_{cr} \simeq 173$ is a critical value at which the lowest energy-level of the bound state $1S_{1/2}$ encounters the negative energy continuum, while other bound states $2P_{1/2}, 2S_{3/2}, \dots$ encounter the negative energy continuum at $Z_{cr} > 173$, thus energy-level crossings and productions of electron and positron pair takes place, provided these bound states are unoccupied. We refer the readers to [11–19] for mathematical and numerical details.

The energetics of this phenomenon can be understood as follows. The energy-level of the bound state $1S_{1/2}$ can be estimated as follows,

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$$\mathcal{E}(1S_{1/2}) = mc^2 - \frac{Ze^2}{\bar{r}} < -mc^2, \qquad (6)$$

where $\bar{r}$ is the average radius of the $1S_{1/2}$ state's orbit, and the binding energy of this state $Ze^2/\bar{r} > 2mc^2$. If this bound state is unoccupied, the bare nucleus gains a binding energy $Ze^2/\bar{r}$ larger than $2mc^2$, and becomes unstable against the production of an electron–positron pair. Assuming this pair-production occur around the radius $\bar{r}$, we have energies of electron (ϵ_-) and positron (ϵ_+):

$$\epsilon_- = \sqrt{(c|\mathbf{p}_-|)^2 + m^2c^4} - \frac{Ze^2}{\bar{r}};$$
$$\epsilon_+ = \sqrt{(c|\mathbf{p}_+|)^2 + m^2c^4} + \frac{Ze^2}{\bar{r}}, \qquad (7)$$

where $\mathbf{p}_\pm$ are electron and positron momenta, and $\mathbf{p}_- = -\mathbf{p}_+$. The total energy required for a pair production is,

$$\epsilon_{-+} = \epsilon_- + \epsilon_+ = 2\sqrt{(c|\mathbf{p}_-|)^2 + m^2c^4}, \qquad (8)$$

which is independent of the potential $V(\bar{r})$. The potential energies $\pm eV(\bar{r})$ of electron and positron cancel each other and do not contribute to the total energy (8) required for pair production. This energy (8) is acquired from the binding energy ($Ze^2/\bar{r} > 2mc^2$) by the electron filling into the bound state $1S_{1/2}$. A part of the binding energy becomes the kinetic energy of positron that goes out. This is analogous to the familiar case that a proton ($Z = 1$) catches an electron into the ground state $1S_{1/2}$, and a photon is emitted with the energy not less than 13.6 eV.

In this Letter, we study classical and semi-classical states of electrons, electron–positron pair production in an electric potential of macroscopic cores with charge $Q = Z|e|$, mass M and macroscopic radius R_c.

2. Classical description of electrons in potential of cores

2.1. Effective potentials for particle's radial motion

Setting the origin of spherical coordinates (r, θ, ϕ) at the center of such cores, we write the vectorial potential $A_\mu = (\mathbf{A}, A_0)$, where $\mathbf{A} = 0$ and A_0 is the Coulomb potential. The motion of a relativistic electron with mass m and charge e is described by its radial momentum p_r, total angular momenta p_ϕ and the Hamiltonian,

$$H_\pm = \pm mc^2\sqrt{1 + \left(\frac{p_r}{mc}\right)^2 + \left(\frac{p_\phi}{mcr}\right)^2} - V(r), \qquad (9)$$

where the potential energy $V(r) = eA_0$, and $\pm$ corresponds for positive and negative energies. The states corresponding to negative energy solutions are fully occupied. The total angular momentum p_ϕ is conserved, for the potential $V(r)$ is spherically symmetric. For a given angular momentum $p_\phi = mv_\perp r$, where $v_\perp$ is the transverse velocity, the effective potential energy for electron's radial motion is

$$E_\pm(r) = \pm mc^2\sqrt{1 + \left(\frac{p_\phi}{mcr}\right)^2} - V(r), \qquad (10)$$

where $\pm$ indicates positive and negative effective energies, outside the core ($r \geqslant R_c$), the Coulomb potential energy $V(r)$ is given by

$$V_{\rm out}(r) = \frac{Ze^2}{r}. \qquad (11)$$

Inside the core ($r \leqslant R_c$), the Coulomb potential energy is given by

$$V_{\rm in}(r) = \frac{Ze^2}{2R_c}\left[3 - \left(\frac{r}{R_c}\right)^2\right], \qquad (12)$$

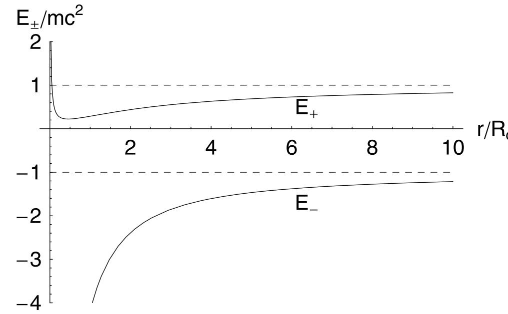

Fig. 1. In the case of point-like charge distribution, we plot the positive and negative effective potential energies $E_\pm$ (10), $p_\phi/(mcR_c) = 2$ and $Ze^2 = 1.95mc^2R_c$, to illustrate the radial location R_L (14) of stable orbits where E_+ has a minimum (15). All stable orbits are described by $cp_\phi > Ze^2$. The last stable orbits are given by $cp_\phi \to Ze^2 + 0^+$, whose radial location $R_L \to 0$ and energy $\mathcal{E} \to 0^+$. There is no any stable orbit with energy $\mathcal{E} < 0$ and the energy-level crossing with the negative energy spectrum E_- is impossible.

where we postulate the charged core has a uniform charge distribution with constant charge density $\rho = Ze/V_c$, and the core volume $V_c = 4\pi R_c^3/3$. Coulomb potential energies outside the core (11) and inside the core (12) are continuous at $r = R_c$. The electric field on the surface of the core,

$$E_s = \frac{Q}{R_c^2} = \frac{\lambda_e}{R_c}E_c, \qquad \beta \equiv \frac{Ze^2}{mc^2R_c} \qquad (13)$$

where the electron Compton wavelength $\lambda_e = \hbar/(mc)$, the critical electric field $E_c = m^2c^3/(e\hbar)$ and the parameter β is the electric potential-energy on the surface of the core in unit of the electron mass-energy.

2.2. Stable classical orbits (states) outside the core

Given different values of total angular momenta p_ϕ, the stable circulating orbits R_L (states) are determined by the minimum of the effective potential $E_+(r)$ (10) (see Fig. 1), at which $dE_+(r)/dr = 0$. We obtain stable orbits locate at the radii R_L outside the core,

$$R_L = \left(\frac{p_\phi^2}{Ze^2m}\right)\sqrt{1 - \left(\frac{Ze^2}{cp_\phi}\right)^2}, \quad R_L \geqslant R_c, \qquad (14)$$

for different p_ϕ-values. Substituting Eq. (14) into Eq. (10), we find the energy of electron at each stable orbit,

$$\mathcal{E} \equiv \min(E_+) = mc^2\sqrt{1 - \left(\frac{Ze^2}{cp_\phi}\right)^2}. \qquad (15)$$

For the condition $R_L \gtrsim R_c$, we have

$$\left(\frac{Ze^2}{cp_\phi}\right)^2 \lesssim \frac{1}{2}\left[\beta(4 + \beta^2)^{1/2} - \beta^2\right], \qquad (16)$$

where the semi-equality holds for the last stable orbits outside the core $R_L \to R_c + 0^+$. In the point-like case $R_c \to 0$, the last stable orbits are

$$cp_\phi \to Ze^2 + 0^+, \qquad R_L \to 0^+, \qquad \mathcal{E} \to 0^+. \qquad (17)$$

Eq. (15) shows that there are only positive or null energy solutions (states) in the case of a point-like charge, which corresponds to the energy spectra equations (3), (4), (5) in quantum mechanic scenario. While for $p_\phi \gg 1$, radii of stable orbits $R_L \gg 1$ and energies $\mathcal{E} \to mc^2 + 0^-$, classical electrons in these orbits are critically bound for their banding energy goes to zero. We conclude that the energies (15) of stable orbits outside the core must be smaller