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A dispersion relation for the pion-mass dependence of hadron properties

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ABSTRACT

We present a dispersion relation in the pion-mass squared, which static quantities (nucleon mass, magnetic moment, etc.) obey under the assumption of analyticity in the entire complex m_π^2 plane modulo a cut at negative m_π^2 associated with pion production. The relation is verified here in a number of examples of nucleon and Δ-isobar properties computed in chiral perturbation theory up to order p^3. We outline a method to obtain relations for other mass-dependencies and illustrate it on a two-loop example.

1. Introduction

Present lattice QCD calculations are still limited to larger than physical values of light quark masses, $m_q > m_{u,d} \simeq 5$–10 MeV, but the chiral perturbation theory (χPT) [1,2] can, in many cases, be applied to bridge the gap between the lattice and the real world (see, e.g., [3–8]). χPT can predict at least some of the 'non-analytic' dependencies of static quantities (masses, magnetic moments, etc.) on pion-mass squared, or the quark mass ($m_\pi^2 \sim m_q$). The rest of the contributions contain the *a priory* unknown low-energy constants (LECs). In this Letter we examine the origins of non-analytic dependencies arising in χPT, by considering analytic properties of the chiral expansion in the complex m_π^2 plane.

The basic observation is that chiral loops exhibit a cut along the *negative* m_π^2 axis. The cut is associated with pion production which can occur without any excess of energy for $m_\pi^2 \leqslant 0$. Assuming analyticity in the rest of the m_π^2-plane (see Fig. 1), one arrives at a dispersion relation of the type:

$$f(m_\pi^2) = -\frac{1}{\pi}\int_{-\infty}^{0} dt\, \frac{\operatorname{Im} f(t)}{t - m_\pi^2 + i0^+}, \tag{1}$$

where f is a static quantity, 0^+ is an infinitesimally small positive number. In what follows, we explicitly verify this type of dispersion relation on a few examples of the nucleon and $\Delta(1232)$-isobar properties and discuss its field of application. In particular, we consider a two-loop example (a sunset graph) for which the absorptive part can be extracted relatively easy. We conclude by comparing this dispersion relation with a similar "mass-dispersion" relation long-known in the literature.

2. Nucleon mass

We begin right away by considering the *nucleon* properties as a function of $t = m_\pi^2$. For example, the pion-mass dependence of the nucleon mass, computed to the nth order in the chiral expansion, can be written as:

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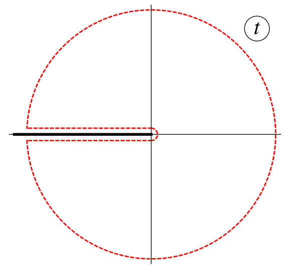

Fig. 1. The cut and the contour in the complex $t = m_\pi^2$ plane, which go into the derivation of the dispersion relation in Eq. (1).

$$M_N = \sum_{\text{even}\,\ell}^{n} a_\ell t^{\frac{\ell}{2}} + \sum_{\ell}^{n} \Sigma_N^{(\ell)}(t), \tag{2}$$

where a's are some linear combinations of LECs, $\Sigma_N^{(\ell)}(t)$ is the ℓth order nucleon self-energy given by the graphs of the type shown in Fig. 2. According to the power counting rules [9], a graph with L loops, N_π pion and N_N nucleon lines, V_k vertices from the Lagrangian of order k, contributes at order p^n, with p being the generic light scale and

$$n = \sum_k k V_k + 4L - 2N_\pi - N_N. \tag{3}$$

The leading order pion–nucleon Lagrangian is of order $k = 1$, and, to the first order in the pion-field $\pi^a(x)$ (with index $a = 1, 2, 3$), is written as [10]:

$$\mathcal{L}_{\pi N}^{(1)} = \bar{N}(x)\left(i\partial\!\!\!/ - \mathring{M}_N + \frac{\mathring{g}_A}{2\,\mathring{f}_\pi}(\partial\!\!\!/\pi^a)\tau^a\gamma_5\right)N(x) + \text{c.t.} + \mathcal{O}(\pi^2), \tag{4}$$

where $N(x)$ is the isospin-doublet nucleon field, τ^a are Pauli matrices, $\mathring{M}_N$, $\mathring{g}_A$, and $\mathring{f}_\pi$ are respectively: the nucleon mass, axial-coupling and pion-decay constants, in the chiral limit ($m_\pi \to 0$); "c.t." stands for counter-term contributions, which are required for the renormalization of the nucleon mass, field, and so on.

The self-energy receives its leading contribution at order p^3, which is given by the graph Fig. 2(a) and the following expression:

$$\Sigma_N^{(3)}(t) = \frac{3g_A^2}{4f_\pi^2} i \int \frac{d^4k}{(2\pi)^4} \frac{k\cdot\gamma\gamma_5(p\cdot\gamma - k\cdot\gamma + M_N)k\cdot\gamma\gamma_5}{(k^2 - t + i0^+)[(p-k)^2 - M_N^2 + i0^+]}\bigg|_{p\cdot\gamma = M_N} \tag{5a}$$

$$\overset{\text{dim reg}}{=} -\frac{3g_A^2}{4f_\pi^2}\frac{M_N^3}{(4\pi)^2}\int_0^1 dx\,\big\{\big[x^2 + (1-x)\tau\big]\big(L_\varepsilon + \ln\big[x^2 + (1-x)\tau - i0^+\big]\big) + \big[2x^2 - (2+x)\tau\big]\big(L_\varepsilon + 1 + \ln\big[x^2 + (1-x)\tau - i0^+\big]\big) - 3L_\varepsilon\big\}, \tag{5b}$$

where $\tau = t/M_N^2$, $L_\varepsilon = -1/\varepsilon - 1 + \gamma_E - \ln(4\pi\Lambda/M_N)$ exhibits the ultraviolet (UV) divergence as $\varepsilon = (4-d)/2 \to 0$, with d being the number of space–time dimensions, Λ the scale of dimensional regularization, and $\gamma_E \simeq 0.5772$ the Euler's constant. Note that for simplicity we assume the physical values for the parameters: $M_N \simeq 939$ MeV, $g_A \simeq 1.267$, $f_\pi \simeq 92.4$ MeV; the difference with the chiral-limit values leads to higher order effects.

After integration over the Feynman parameter x, this result can be written as:

$$\Sigma_N^{(3)}(t) = \frac{3g_A^2 M_N^3}{2(4\pi f_\pi)^2}\left\{-L_\varepsilon + (1 - L_\varepsilon)\frac{t}{M_N^2}\right\} + \overline{\Sigma}_N^{(3)}, \tag{6a}$$

$$\text{with}\quad \overline{\Sigma}_N^{(3)}(t) = -\frac{3g_A^2 M_N^3}{(4\pi f_\pi)^2}\left(\tau^{3/2}\sqrt{1 - \frac{1}{4}\tau}\arccos\left(\frac{1}{2}\sqrt{\tau}\right) + \frac{1}{4}\tau^2\ln\tau\right). \tag{6b}$$

The term in figure brackets, containing the UV-divergence, must be entirely canceled by the counter-term contribution [11], which, to this order, is of the form: $a_0 + a_2 t$, where a's contain the "bare" values of the LECs. The first term in brackets can be viewed as a renormalization of the nucleon mass, while the second as a renormalization of the πN sigma term. The remaining part, $\overline{\Sigma}_N$, is UV-finite and consistent with the power counting in the sense that its size is indeed of order p^3.

Let us now see whether this contribution obeys the dispersion relation of the type stated in Eq. (1). The imaginary part can be easily found from Eq. (5b) by taking into account that $\ln(-1 + i0^+) = i\pi$,

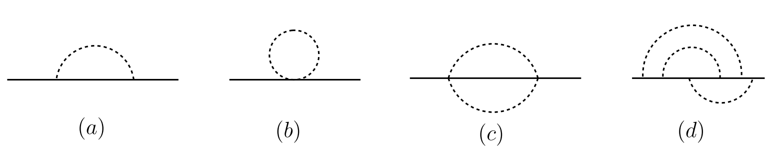

Fig. 2. Graphs representing chiral-loop corrections to the nucleon mass. Nucleon (pion) propagators are denoted by solid (dashed) lines.