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Nonabelian parafermions and their dimensions

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A B S T R A C T

We propose a generalization of the Zamolodchikov–Fateev parafermions which are abelian, to nonabelian groups. The fusion rules are given by the tensor product of representations of the group. Using Vafa equations we get the allowed dimensions of the parafermions. We find for simple groups that the dimensions are integers. For cover groups of simple groups, we find, for $n.G.m$, that the dimensions are the same as Z_n parafermions. Examples of integral parafermionic systems are studied in detail.

Conformal field theory in two dimensions has been a source of numerous results owing to its solvability and its rich structure. It has been successfully applied to statistical mechanics and string theory.

One line of such ideas is a generalization of the Ising model to so-called parafermions, first put forward by Zamolodchikov and Fateev [1]. These are analytic currents with nonintegral spin. The known examples, so far, are based on the cyclic group Z_n, or products of cyclic groups, that is, abelian groups. The parafermions appear in nature as fixed points of some models of magnets, e.g., the Andrews Baxter Forrester models [2] and Z_n clock models [1]. Also, they are vital components in string compactification, since they are closely related to $N=2$ superconformal theories, yielding solvable realistic string theories, featuring for example, in the models of Ref. [3].

Our idea is to generalize the parafermions to nonabelian groups. We can write a parafermionic system for any group G. This we do by assuming that the currents fall into representations of the group G. I.e., we have a parafermionic multiplet for each representation of the group, G, and for each vector of the representation. We then postulate that the OPE of the parafermionic system obey the group symmetry. This is a direct generalization of the notion of parafermions and, in the abelian case, it gives the usual results of Zamolodchikov and Fateev.

To be specific, assume that I ranges over the representations of the group G and that the index i ranges over the vectors in each representation. We introduce a parafermion which is a field $\psi_i^I(z)$, which is an a holomorphic field of dimension Δ_I. Let f_{ijk}^{IJK} denote the Clebsh–Gordon coefficient of the group. I.e., for each element of the group $g \in G$ we have the relation,

$$\sum_{i',j',k'} \Phi_{ii'}^I(g)\Phi_{jj'}^J(g)\Phi_{kk'}^K(g) f_{i'j'k'}^{IJK} = f_{ijk}^{IJK}, \tag{1}$$

where $\Phi_{ii'}^I(g)$ is a matrix in the Ith representation of the group G.

We further define,

$$f_{ij1}^{IJ1} = b_{ij}^{IJ}, \tag{2}$$

and

$$\sum_k b_{k'k}^{K\bar{K}} f_{ij}^{k(IJ\bar{K})} = f_{ijk'}^{IJK}. \tag{3}$$

We then postulate the following operator product expansions (OPE) for the parafermions,

$$\begin{aligned}\psi_i^I(z)\psi_j^J(w) = {} & \frac{b_{ij}^{IJ} C_I}{(z-w)^{2\Delta_I}} \\ & + \sum_{k,K} f_{ij}^{k(IJK)} (z-w)^{-\Delta_I-\Delta_J+\Delta_K} \\ & \times \left[\tilde{C}_K^{IJ}\psi_k^K(w) + \tilde{\tilde{C}}_K^{IJ}(z-w)\partial\psi_k^K(w)\right] \\ & + \text{h.o.t.},\end{aligned} \tag{4}$$

where h.o.t. stands for higher order terms. The constants C, $\tilde{C}$ and $\tilde{\tilde{C}}$ are determined by the associativity of the OPE above, once the dimensions Δ_I are determined. We postulate also $\psi_1^1(z) = T(z)$, the stress tensor, $\Delta_1 = 2$, $C_1 = c/2$, where c is the central charge.

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Thus the algebra contains the Virasoro algebra and we demand, accordingly, that, $\psi_i^I(z)$ is a primary field, $\tilde{C}_I^{1I} = \Delta_I$, and, $\tilde{\tilde{C}}_I^{1I} = 1$.

The fusion rules, i.e., the way operators fuse in the operator product expansion, are then given simply by the tensor product algebra of the representations of the group. This is easily calculated in specific examples by means of the character tables of specific groups. Denote by $\chi_I(g)$ the character in the representation I of the group, $g \in G$,

$$\chi_I(g) = \sum_i \Phi_{ii}^I(g). \tag{5}$$

Then the fusion coefficients of the parafermions are given by,

$$N_{IJ}^K = \frac{1}{O(G)} \sum_{g\in G} \chi_I(g)\chi_J(g)\chi_K(g)^*, \tag{6}$$

which is reminiscent of the Verlinde formula [4]. Here, $O(G)$ is the order of the group.

Thus, we can use Vafa's equations [5] to calculate the dimensions of the parafermions, which are found up to some arbitrary integer multiplicative factor. Denote by

$$\alpha_I = e^{2\pi i \Delta_I}. \tag{7}$$

Then Vafa equations are

$$(\alpha_I\alpha_J\alpha_K\alpha_L)^{N_{IJKL}} = \prod_R \alpha_R^{N_{IJKL;R}}, \tag{8}$$

where

$$N_{IJKL} = \sum_R N_{IJR} N_{KL}^R, \tag{9}$$

and

$$N_{IJKL;R} = N_{IJR}N_{KL}^R + N_{IKR}N_{JL}^R + N_{ILR}N_{JK}^R, \tag{10}$$

where we define $N_{IJK} = N_{IJ}^{\bar{K}}$.

When G is abelian, we are back in the case of Zamolodchikov and Fateev. For a $G = Z_N$ group, we denote the I parafermion for the representation $\Phi_I(e^{2\pi i r/N}) = e^{2\pi i r I/N}$, for any I and r modulo N. The dimension of the Ith parafermion is Δ_I and $\Delta_I = \Delta_{N-I}$ since it is the complex conjugate field. We find from Eq. (6) that the structure constant is $N_{IJ}^K = 1$ if $K - I - J = 0 \bmod N$ and is zero otherwise. Here Vafa's equations become,

$$\Delta_I + \Delta_J + \Delta_K + \Delta_L = \Delta_{K+L} + \Delta_{K+I} + \Delta_{K+J} \bmod Z, \tag{11}$$

where I, J, K, L are any integers modulo N such that $I + J + K + L = 0 \bmod N$. This equation, already appears in Ref. [1], Eq. (A4) there, derived from the mutual semilocality. This Eq. (11) implies, in particular, by taking $I = J = 1$, that

$$2\Delta_{K+1} - \Delta_K - \Delta_{K+2} = \beta \bmod Z, \tag{12}$$

where $\beta = 2\Delta_1 - \Delta_2$. Thus, $\Delta_K = -\beta K^2/2 \bmod Z$ is the unique solution to Eq. (11), which satisfies, $\Delta_0 =$ integer and $\Delta_1 = \Delta_{N-1}$. It follows that

$$\Delta_I = M_I + mI^2/(sN), \tag{13}$$

where M_I and m are arbitrary integers and $s = 1$ for odd N and $s = 2$ for even N. We set $\Delta_r = \Delta_{N-r}$. Thus, this method is consistent with the known abelian case.

Thus, for each group we simply substitute the characters into Vafa equations to find the dimensions of the parafermions.

Let us introduce some basic notions of group theory. The group G is called simple if the only normal subgroups are itself or the trivial one. An automorphism is a one to one and onto map $\sigma : G \to G$ such that

$$\sigma(gh) = \sigma(g)\sigma(h), \tag{14}$$

where $g, h \in G$. An internal automorphism is the map,

$$\sigma_h(g) = hgh^{-1}, \tag{15}$$

where h is a fixed element of the group. We denote the automorphism group by $\mathrm{Aut}(G)$, which is a group under decompositions. We denote by $\mathrm{Int}(G)$ the internal automorphism subgroup of $\mathrm{Aut}(G)$, which is a normal subgroup. The outer automorphism group, $\mathrm{Out}(G)$ is defined as the quotient group,

$$\mathrm{Out}(G) = \frac{\mathrm{Aut}(G)}{\mathrm{Int}(G)}. \tag{16}$$

The group G itself is a subgroup of $\mathrm{Aut}(G)$ by identifying it with $\mathrm{Int}(G)$ (we assume that G is centerless, see below. If we denote the center by $Z(G)$ then, $\mathrm{Int}(G) \approx G/Z(G)$). A group, H, is called almost simple, if there exists a simple group G such that

$$G \subset H \subset \mathrm{Aut}(G). \tag{17}$$

We call H a cover group by automorphism of the group G.

We call the parafermion system integral if the only solution to Vafa equation is that all the dimensions are integral. Our result about this can be phrased as:

Conjecture 1. *The parafermion system of the group H is integral if and only if H is a nonabelian almost simple group or the trivial group.*

Another type of cover group is an extension by a center. The center of a group H is the subgroup of elements that commute with every group member. The center of the group H is a normal subgroup denoted by $Z(H)$, $h \in Z(H)$ if and only if, $gh = hg$ for all $g \in H$. Of course the center of a nonabelian simple group is trivial. If H is a group such that $G = H/Z(H)$, where G is a simple group, we call H a cover group of G by a center. Other type of center group is mixed both by an automorphism and by a center, where $W = H/Z(H)$ is an almost simple group of the simple group G, $G \subset W \subset \mathrm{Aut}(G)$. We denote such a cover group as $H = n.G.m$, where the center is a Z_n group and the outer automorphism is a Z_m group, i.e., the quotient of the almost simple group to its simple subgroup is a Z_m group. Of course, the outer automorphisms group can be nonabelian (for a simple group it is always solvable).

Now, suppose that H is a Z_n cover group of an almost simple group G, $G = H/Z(H)$, where $Z(H) \approx Z_n$. Let I be an irreducible representation of H. By Schur's lemma, since the center commutes with all elements of H, it is an irreducible representation of Z_n. These representations are denoted by their charge modulo n. So let $d_I \bmod n$ be the charge of the Ith representation. Our result can then be quoted as:

Conjecture 2. *The parafermion system of a cover group of the type $H = n.G.m$, where G is a nonabelian simple group, gives the same dimensions as Z_n parafermions. I.e., the dimension of the Ith representation is,*

$$\Delta_I = M_I + md_I^2/(sn), \tag{18}$$

where $s = 1$ ($s = 2$) for odd (even) n, and M_I and m are some integers.

The same applies to a center which is a product of cyclic groups. One just adds up the dimensions.