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Infinitely many shape invariant potentials and new orthogonal polynomials

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ABSTRACT

Three sets of exactly solvable one-dimensional quantum mechanical potentials are presented. These are shape invariant potentials obtained by deforming the radial oscillator and the trigonometric/hyperbolic Pöschl–Teller potentials in terms of their degree ℓ polynomial eigenfunctions. We present the entire eigenfunctions for these Hamiltonians ($\ell = 1, 2, \ldots$) in terms of new orthogonal polynomials. Two recently reported shape invariant potentials of Quesne and Gómez-Ullate et al.'s are the first members of these infinitely many potentials.

1. Introduction

In this Letter we present two infinite sets and one finite set of exactly solvable one-dimensional quantum mechanical Hamiltonians. As the main part of the eigenfunctions, a new type of orthogonal polynomials is obtained for each Hamiltonian. They are exactly solvable by combining shape invariance [1] with the factorisation method [2,3] or the so-called supersymmetric quantum mechanics [4]. Then the entire energy spectrum and the corresponding eigenfunctions can be obtained algebraically. However, these new shape invariant Hamiltonians do not possess the exact Heisenberg operator solutions [5], in contrast to most of the known shape invariant Hamiltonians.

Shape invariance is a sufficient condition for exactly solvable quantum mechanical systems. Based on one shape invariant potential, an infinite number of exactly solvable potentials and their eigenfunctions can be constructed by a modification of Crum's method [6,7]. But these newly derived systems fail to inherit the shape invariance, nor do they possess Heisenberg operator solutions. Although several shape invariant 'discrete' quantum mechanical systems are added to recently [8], the catalogue of the shape invariant potentials was rather short for a long time. In 2008, Quesne [9] reported two new shape invariant potentials based on the Sturm–Liouville problems for the X_1-Laguerre and the X_1-Jacobi polynomials proposed by Gómez-Ullate et al. [10].

Here we present our preliminary results on the three sets of shape invariant potentials and the corresponding new types of orthogonal polynomials, without proof. After brief introduction of notation and the shape invariance method, they are obtained by *deforming* the well-known shape invariant potentials, the radial oscillator and the Darboux–Pöschl–Teller [11,12] potentials, in terms of the degree ℓ polynomial eigenfunctions, *i.e.* the Laguerre and the Jacobi polynomials. The eigenpolynomials of the new Hamiltonians are orthogonal polynomials starting from degree ℓ, which could be called X_ℓ polynomials. The Quesne–Gómez-Ullate et al. examples [9,10] correspond to the $\ell = 1$ cases.

2. General setting: Shape invariance

The starting point is a generic one-dimensional quantum mechanical system having a square-integrable groundstate together with a finite or infinite number of discrete energy levels: $0 = \mathcal{E}_0 < \mathcal{E}_1 < \mathcal{E}_2 < \cdots$. The groundstate energy $\mathcal{E}_0$ is chosen to be zero, by adjusting the constant part of the Hamiltonian. The *positive semi-definite* Hamiltonian is expressed in a factorised form [2–4]:

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$$\mathcal{H}=\mathcal{A}^\dagger\mathcal{A}=p^2+U(x),\quad p=-i\partial_x, \tag{1}$$

$$\mathcal{A}\stackrel{\text{def}}{=}\partial_x-w'(x),\qquad \mathcal{A}^\dagger=-\partial_x-w'(x), \tag{2}$$

$$U(x)\stackrel{\text{def}}{=}w'(x)^2+w''(x). \tag{3}$$

For simplicity of presentation we have adopted the unit system in which $\hbar$ and the mass m of the particle are such that $\hbar=2m=1$. Here we call a real and smooth function $w(x)$ a *prepotential* and it parametrises the groundstate wavefunction $\phi_0(x)$, which has *no node* and can be chosen real and positive, $\phi_0(x)=e^{w(x)}$. It is trivial to verify $\mathcal{A}\phi_0(x)=0$ and $\mathcal{H}\phi_0(x)=0$.

Shape invariance, a sufficient condition for exact solvability [1], is realised by specific dependence of the potential, or the prepotential on a set of parameters $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots)$, to be denoted by $w(x;\boldsymbol{\lambda})$, $\mathcal{A}(\boldsymbol{\lambda})$, $\mathcal{H}(\boldsymbol{\lambda})$, $\mathcal{E}_n(\boldsymbol{\lambda})$, etc. The shape invariance condition to be discussed in this Letter is

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger=\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})+\mathcal{E}_1(\boldsymbol{\lambda}), \tag{4}$$

$$\begin{aligned}&w'(x;\boldsymbol{\lambda})^2-w''(x;\boldsymbol{\lambda})\\&\quad=w'(x;\boldsymbol{\lambda}+\boldsymbol{\delta})^2+w''(x;\boldsymbol{\lambda}+\boldsymbol{\delta})+\mathcal{E}_1(\boldsymbol{\lambda}),\end{aligned} \tag{5}$$

in which $\boldsymbol{\delta}$ is a certain shift of the parameters. Then the entire set of discrete eigenvalues and the corresponding eigenfunctions of $\mathcal{H}=\mathcal{H}(\boldsymbol{\lambda})$

$$\mathcal{H}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda})=\mathcal{E}_n(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}) \tag{6}$$

is determined algebraically [1,4,8]:

$$\mathcal{E}_n(\boldsymbol{\lambda})=\sum_{k=0}^{n-1}\mathcal{E}_1(\boldsymbol{\lambda}+k\boldsymbol{\delta}), \tag{7}$$

$$\begin{aligned}\phi_n(x;\boldsymbol{\lambda})&\propto\mathcal{A}(\boldsymbol{\lambda})^\dagger\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\cdots\mathcal{A}\big(\boldsymbol{\lambda}+(n-1)\boldsymbol{\delta}\big)^\dagger\\&\quad\times e^{w(x;\boldsymbol{\lambda}+n\boldsymbol{\delta})}.\end{aligned} \tag{8}$$

3. The radial oscillator

Here we present an infinite number of shape invariant potentials indexed by a non-negative integer $\ell=0,1,2,\ldots$. For $\ell=0$, it is the well-known radial oscillator, or the harmonic oscillator with a centrifugal barrier potential, with $\boldsymbol{\lambda}=g>0$:

$$\mathcal{H}_0(g)=p^2+x^2+\frac{g(g-1)}{x^2}-1-2g, \tag{9}$$

$$w_0(x;g)=-\frac{1}{2}x^2+g\log x,\quad 0<x<\infty. \tag{10}$$

Here we adopt the notation of our previous work [5, §III.A.1]. The shape invariance, the Heisenberg operator solution and the creation–annihilation operators of the above Hamiltonian are discussed in some detail there. It is trivial to verify (5) with $\boldsymbol{\delta}=1$, $\mathcal{E}_1(g)=4$ and we obtain the equidistant spectrum and the corresponding eigenfunctions $n=0,1,2,\ldots,$

$$\mathcal{E}_n(g)=4n, \tag{11}$$

$$\phi_n(x;g)=P_n\big(x^2;g\big)e^{w_0(x;g)},\quad P_n(x;g)=L_n^{(g-\frac{1}{2})}(x). \tag{12}$$

The polynomial eigenfunctions are the Laguerre polynomials in x^2, which are orthogonal with respect to the measure $\phi_0(x)^2=e^{2w_0(x;g)}=e^{-x^2}x^{2g}$.

For each positive integer $\ell\geqslant 1$, let us introduce a prepotential and a Hamiltonian:

$$\xi_\ell(x;g)\stackrel{\text{def}}{=}L_\ell^{(g+\ell-\frac{3}{2})}(-x), \tag{13}$$

$$w_\ell(x;g)\stackrel{\text{def}}{=}w_0(x;g+\ell)+\log\frac{\xi_\ell(x^2;g+1)}{\xi_\ell(x^2;g)}, \tag{14}$$

$$\mathcal{A}_\ell(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\partial_x-w_\ell'(x;\boldsymbol{\lambda}),\qquad \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger=-\partial_x-w_\ell'(x;\boldsymbol{\lambda}), \tag{15}$$

$$\mathcal{H}_\ell(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\mathcal{A}_\ell(\boldsymbol{\lambda}). \tag{16}$$

Since the polynomial $\xi_\ell(x^2;g)$ has no zero in the domain $0<x<\infty$, the prepotential and the potential are smooth in the entire domain. It is straightforward to verify the shape invariance condition (5) with $\boldsymbol{\delta}=1$, $\mathcal{E}_{\ell,1}(g)=4$. By using (8) as a Rodrigues type formula, we obtain the complete set of eigenfunctions with the equidistant spectrum:

$$\mathcal{H}_\ell(g)\phi_{\ell,n}(x;g)=\mathcal{E}_{\ell,n}(g)\phi_{\ell,n}(x;g),\quad n=0,1,\ldots, \tag{17}$$

$$\mathcal{E}_{\ell,n}(g)=\mathcal{E}_n(g+\ell)=4n, \tag{18}$$

$$\phi_{\ell,n}(x;g)=P_{\ell,n}\big(x^2;g\big)\psi_\ell(x),\quad \psi_\ell(x)\stackrel{\text{def}}{=}\frac{e^{w_0(x;g+\ell)}}{\xi_\ell(x^2;g)}, \tag{19}$$

$$\begin{aligned}P_{\ell,n}(x;g)&\stackrel{\text{def}}{=}\xi_\ell(x;g+1)P_n(x;g+\ell)\\&\qquad-\xi_{\ell-1}(x;g+2)P_{n-1}(x;g+\ell).\end{aligned} \tag{20}$$

Obviously we have $P_{\ell,0}(x;g)=\xi_\ell(x;g+1)$ and $\phi_{\ell,0}(x;g)=e^{w_\ell(x;g)}$. The polynomial eigenfunction $P_{\ell,n}(x^2;g)$ is a degree $\ell+n$ polynomial in x^2 but it has only n zeros in the domain $0<x<\infty$. These polynomials are orthogonal with respect to the measure $\psi_\ell(x;g)^2$:

$$\begin{aligned}&\int_0^\infty dx\,\psi_\ell(x;g)^2P_{\ell,n}\big(x^2;g\big)P_{\ell,m}\big(x^2;g\big)\\&\quad=\frac{1}{2n!}\bigg(n+g+2\ell-\frac{1}{2}\bigg)\Gamma\bigg(n+g+\ell-\frac{1}{2}\bigg)\delta_{nm}.\end{aligned} \tag{21}$$

They form a complete basis of the Hilbert space just like the Laguerre polynomials in the $\ell=0$ case. These new types of polynomials do not satisfy the three term recurrence relation, a characteristic feature of all the ordinary orthogonal polynomials. It should be stressed that all four terms in (20) are the Laguerre polynomials of the same index, $g+\ell-1/2$. The action of the operators $\mathcal{A}_\ell(g)$ and $\mathcal{A}_\ell(g)^\dagger$ on the eigenfunctions are:

$$\begin{aligned}&\mathcal{A}_\ell(g)\phi_{\ell,n}(x;g)=-2\phi_{\ell,n-1}(x;g+1),\\&\mathcal{A}_\ell(g)^\dagger\phi_{\ell,n-1}(x;g+1)=-2n\phi_{\ell,n}(x;g).\end{aligned} \tag{22}$$

For $\ell=1$ the Hamiltonian reads

$$\begin{aligned}\mathcal{H}_1(g)&=p^2+x^2+\frac{g(g+1)}{x^2}-3-2g\\&\qquad+\frac{4}{x^2+g+\frac{1}{2}}-\frac{4(2g+1)}{(x^2+g+\frac{1}{2})^2},\end{aligned}$$

which is equivalent to that of the shape invariant potential of Quesne Eq. (8) of [9] with the replacement $\omega\to 2$ and $l\to g$. The formula (20) expressing the polynomial eigenfunctions in terms of the Laguerre polynomials is the generalisation of Gómez-Ullate et al.'s [10] relation Eq. (80) between the X_1-Laguerre and the Laguerre polynomials.

4. Darboux–Pöschl–Teller potential

Here we present another infinite number of shape invariant potentials indexed by a non-negative integer $\ell=0,1,2,\ldots$. For