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Stability of asymmetric tetraquarks in the minimal-path linear potential

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ABSTRACT

The linear potential binding a quark and an antiquark in mesons is generalized to baryons and multiquark configurations as the minimal length of flux tubes neutralizing the color, in units of the string tension. For tetraquark systems, i.e., two quarks and two antiquarks, this involves the two possible quark–antiquark pairings, and the Steiner tree linking the quarks to the antiquarks. A novel inequality for this potential demonstrates rigorously that within this model the tetraquark is stable in the limit of large quark-to-antiquark mass ratio.

The quark–antiquark confinement in ordinary mesons is often described by a linear potential $V_2 = r$, in units where the string tension is set to unity. For a given interquark separation r, it can be interpreted as the minimal gluon energy if the field is localized in a flux tube of constant section linking the quark to the antiquark.

The natural extension to describe the confinement of three quarks in a baryon is the so-called Y-shape potential

$$V_3(v_1, v_2, v_3) = \min_s(d_1 + d_2 + d_3), \tag{1}$$

where d_i is the distance of the ith quark located at v_i $(i = 1, 2, 3)$ to a junction s whose location is adjusted to minimize V_3. This potential has been proposed in Refs. [1–7], among others. It has been used, e.g., in Refs. [8,9] for studying the spectroscopy of baryons. See, also [10]. The optimization in (1) corresponds to the well-known problem of Fermat and Torricelli to link three points with the minimal network. See Fig. 1.

We now turn to the tetraquark systems $(Q, Q, \bar{q}, \bar{q})$, with the notation (v_1, v_2, v_3, v_4) for the locations, and (M, M, m, m) for the masses which will be used shortly. The potential is assumed to be (with $d_{ij} = \|v_i v_j\|$)

$$U = \min\{d_{13} + d_{24}, d_{14} + d_{23}, V_4\},$$
$$V_4 = \min_{s_1, s_2}\bigl(\|v_1 s_1\| + \|v_2 s_1\| + \|s_1 s_2\| + \|s_2 v_3\| + \|s_2 v_4\|\bigr). \tag{2}$$

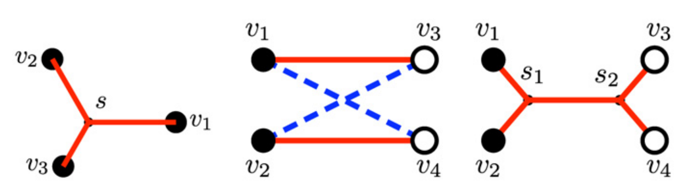

Fig. 1. Generalization of the linear quark–antiquark potential of mesons to baryons (left) and to tetraquarks, where the minimum is taken of the flip–flop (center) and Steiner tree (right) configurations.

The first two terms of U describe the two possible quark–antiquark links, and their minimum is sometimes referred to as the "flip–flop" model, schematically pictured in Fig. 1. It was introduced by Lenz et al. [11], who used, however, a quadratic instead of linear rise of the potential as a function of the distance. The last term, V_4, is represented in Fig. 1 and corresponds to a connected flux tube. It is given by a Steiner tree, i.e., it is minimized by varying the location of the Steiner points s_1 and s_2. The choice of this potential is inspired by Refs. [3,12–14], and has been discussed in the context of lattice QCD [15,16].

The four-body problem in quantum mechanics is notoriously difficult. For instance, Wheeler proposed in 1945 the existence of a positronium molecule (e^+, e^+, e^-, e^-) which is stable in the limit where internal annihilation is neglected, i.e., lies below its threshold for dissociation into two positronium atoms. In 1946, Ore published a four-body calculation of this system [17] and con-

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cluded that his investigation "counsels against the assumption that clusters of this (or even of higher) complexity can be formed". However, in 1947, Hylleraas and the same Ore published an elegant analytic proof that this molecule is stable [18]. It has been discovered recently [19].

Similarly, the above model (2), in its linear version, was considered by Carlson and Pandharipande, who entitled their paper [20] "Absence of exotics in the flux tube model", i.e., did not find stable tetraquarks.[1] However, Vijande et al. [21] used a more systematic variational expansion of the wave function and in their numerical solution of the four-body problem found a stable tetraquark ground state. Moreover, unlike [20], they considered the possibility of unequal masses, and found that stability improves if the quarks are heavier (or lighter) than the antiquarks, in agreement with previous investigations (see, e.g., [21] for references).

It is thus desirable to check whether this minimal-path model supports or not bound states. The present attempt is based on an upper bound on the potential, which leads to an *exactly solvable* four-body Hamiltonian.

With the Jacobi vector coordinates

$$x = v_2 - v_1, \qquad y = v_4 - v_3, \qquad z = \frac{v_3 + v_4 - v_1 - v_2}{2}, \tag{3}$$

and their conjugate momenta, the relative motion is described by the Hamiltonian

$$H = \frac{p_x^2}{M} + \frac{p_y^2}{m} + \frac{p_z^2}{4\mu} + U(x, y, z), \tag{4}$$

where μ, given by $\mu^{-1} = m^{-1} + M^{-1}$, is the quark–antiquark reduced mass. Using the scaling properties of H, one can set $m = 1$ without loss of generality.

The simplest bound on the potential U is

$$U \leqslant V_4 \leqslant \|x\| + \|y\| + \|z\|, \tag{5}$$

as the tree with optimized Steiner points s_1 and s_2 is shorter than if the junctions are set at the middles of the quark separation v_1v_2 and antiquark separation v_3v_4. This leads to a separable upper bound for the Hamiltonian

$$H \leqslant H' = \frac{p_x^2}{M} + \|x\| + p_y^2 + \|y\| + \frac{p_z^2}{4\mu} + \|z\|. \tag{6}$$

Now, the ground state e_0 of $p_x^2 + \|x\|$ corresponds to the radial equation $-u''(r) + ru(r) = e_0 u(r)$ with $u(0) = u(\infty) = 0$ and is the negative of the first zero of the Airy function, $e_0 = 2.3381\ldots$. By scaling, the ground state of $\alpha p_x^2 + \beta\|x\|$, with $\alpha > 0$ and $\beta > 0$ is $\alpha^{1/3}\beta^{2/3}e_0$. Thus the lowest eigenvalue of H' is

$$E' = e_0\left[M^{-1/3} + 1 + (4\mu)^{-1/3}\right], \tag{7}$$

with $\mu = M/(1 + M)$. By comparison, the threshold of $(QQ\bar{q}\bar{q})$ is made of two identical $(Q\bar{q})$ mesons, each governed by the Hamiltonian $h = p^2/(2\mu) + \|r\|$, where p is conjugate to the quark–antiquark separation r. Thus the threshold energy is

$$E_{\text{th}} = 2e_0(2\mu)^{-1/3}, \tag{8}$$

and it is easily seen that $E' > E_{\text{th}}$ for any value of the quark-to-antiquark mass ratio M, i.e., the bound (5) cannot demonstrate binding.

A better bound will be proved below. If there is a genuine Steiner tree[2] linking the quarks to the antiquarks, then

$$V_4 \leqslant \frac{\sqrt{3}}{2}\big(\|x\| + \|y\|\big) + \|z\|. \tag{9}$$

[1] The authors used a relativistic form of kinetic energy and considered also the possibility of short-range corrections, but this seemingly does not affect their conclusion.

[2] This will be made more precise in the proof given in Appendix A.

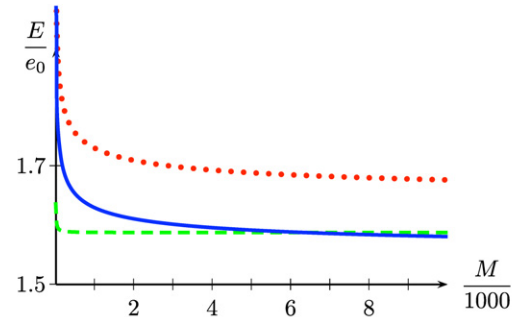

Fig. 2. Simple bound E' (Eq. (7), dotted line) and improved upper bound E'' (Eq. (11), solid line) on the tetraquark ground-state energy as a function of quark-to-antiquark mass ratio M. Also shown is the threshold energy E_{th} (Eq. (8), dashed line). The energies are in units of e_0, the ground state of $-\Delta + \|r\|$.

But if V_4 is not associated to a genuine Steiner tree, this inequality is often violated. Consider for instance a rectangular configuration with $\|v_1v_2\| = \|v_3v_4\| \gg \|v_1v_3\| = \|v_2v_4\|$ (in this case the mathematical Steiner tree problem would require a Steiner point linking v_1 and v_3, another Steiner point linking v_2 and v_4, but the corresponding fluxes are not permitted by the color coupling in QCD), then $\|z\| \sim 0$ and $V_4 \sim \|x\| + \|y\|$, so (9) does not hold.

However, it will be shown that

$$U \leqslant \frac{\sqrt{3}}{2}\big(\|x\| + \|y\|\big) + \|z\|, \tag{10}$$

for any configuration of the quarks and antiquarks, i.e., for any x, y and z. Then the ground state of H is bounded as

$$E < E'' = e_0\left[\left(\frac{3}{4}\right)^{1/3}\big(M^{-1/3} + 1\big) + (4\mu)^{-1/3}\right]. \tag{11}$$

As shown in Fig. 2, this bound E'' significantly improves the previous one, E'. It is easily seen than E'' becomes smaller than E_{th} for very large values of the mass ratio, more precisely for $M > 6402$, and thus that the tetraquark is bound at least in this range of M. The numerical estimate of [21] actually indicates stability for all values of M, even $M = 1$.

To summarize, we obtained an analytic upper bound on the ground state energy of tetraquarks systems with two units of open flavor, $(QQ\bar{q}\bar{q})$, using a model of linear confinement inspired by the strong-coupling regime of QCD. The key is an inequality on the length of a Steiner tree with four terminals. The bound confirms a recent numerical investigation, in which this potential was shown to bind these tetraquarks below the threshold for dissociation into two mesons. It remains to investigate whether this stability survives refinements in the dynamics, such as short range corrections, spin-dependent forces, etc.

It is our intention to extend this investigation to the case of the pentaquark (one antiquark and four quarks) and hexaquark configurations (six quarks), which have been much debated in recent years.

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Appendix A. Results on the Steiner problem

Three terminals. The three-point problem is very much documented in textbooks [22–26]. Let $v_1v_2v_3$ be the triangle, with side