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# Conformal SO(2, 4) transformations of the one-cusp Wilson loop surface

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**Abstract**

By applying the conformal SO(2, 4) transformations to the elementary one-cusp Wilson loop surface we construct various two-cusp and four-cusp Wilson loop surface configurations in $AdS_5$ and demonstrate that they solve the string equations of the Nambu–Goto string action. The conformal boosts of the basic four-cusp Wilson loop surface with a square-form projection generate various four-cusp Wilson loop surfaces with projections of the rescaled square, the rhombus and the trapezium, on which surfaces the classical Euclidean Nambu–Goto string actions in the IR dimensional regularization are evaluated.


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*Keywords:* Bosonic string; Wilson loop surface; Planar gluon amplitude

The AdS/CFT correspondence [1] has more and more revealed the deep relations between the $\mathcal{N}=4$ super-Yang–Mills (SYM) theory and the string theory in $AdS_5 \times S^5$, where classical string solutions play an important role [2–4]. The energies of classical strings have been shown to match with the anomalous dimensions of the gauge invariant operators, while an open string ending on a curve at the boundary of $AdS_5$ has been analyzed to study the strong coupling behavior of the Wilson loop in the gauge theory [5–7].

Alday and Maldacena have used the AdS/CFT correspondence to compute the planar 4-gluon scattering amplitude at strong coupling in the $\mathcal{N}=4$ SYM theory [8] and found agreement with the result of a conjectured form regarding the all-loop iterative structure and the IR divergence of the perturbative gluon amplitude [9]. The 4-gluon scattering amplitude has been evaluated as the string theory computation of the 4-cusp Wilson loop composed of 4 lightlike segments in the T-dual coordinates, where a certain open string solution in $AdS_5$ space is found to minimize the area of the string surface whose boundary conditions are determined by the massless gluon momenta, and a dimensional regularization is used to regularize the IR divergence. Before the IR regularization the worldsheet surface of this particular solution [8] is related by a certain conformal SO(2, 4) transformation to the 1-cusp Wilson loop surface found in [10] (see also [11]).

The non-leading prefactor of the gluon amplitude has been studied [12] and the IR structure of $n$-gluon amplitudes has been fully extracted from a local consideration near each cusp [13], where the 1-cusp Wilson loop solution is constructed even in the presence of the IR regularization. By computing the 1-loop string correction to the 1-cusp Wilson loop solution, the 1-loop coefficient in the cusp anomaly function $f(\lambda)$ of the gauge coupling $\lambda$ has been derived as consistent with the energy of a closed string with large spin $S$ in $AdS_5$ [14]. Moreover, the 2-loop coefficient in $f(\lambda)$ has been presented [15] to agree with the results of [16,17] for the strong coupling solution of the BES equation [18] in the gauge theory side. Based on the string sigma-model action a whole class of string solution for the 4-gluon amplitude has been constructed [19] under the constraint that the Lagrangian evaluated on the string solution takes a constant value. Applying the dressing method [20] used for the study of the giant magnons and their bound or scattering states [21,22] to the elementary 1-cusp Wilson loop solution of [10], new classical solutions for Euclidean worldsheets in $AdS_5$ [23] have been constructed, where the surfaces

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end on complicated, timelike curves at the boundary of $AdS_5$. Several investigations associated with planar gluon amplitudes have been presented [24–30].

In Ref. [8] the planar 4-gluon amplitude at strong coupling has been constructed by deriving the classical string sigma-model action evaluated on the 4-cusp Wilson loop surface whose edge traces out a rhombus on the projected two-dimensional plane at the boundary of $AdS_5$. Based on the Nambu–Goto string action we will apply various conformal SO(2, 4) transformations to the elementary 1-cusp Wilson loop solution of [10]. We will show how the obtained string surface configurations satisfy the string equations of motion derived from the Nambu–Goto string action. We will observe that there appear various kinds of Wilson loop solutions which are separated into the 2-cusp Wilson loop solutions and the 4-cusp ones.

We consider the 1-cusp Wilson loop solution of [10], where the open string world surface ends on two semi infinite lightlike lines and is given by

$$r = \sqrt{2}\sqrt{y_0^2 - y_1^2} \tag{1}$$

embedded in an $AdS_3$ subspace of $AdS_5$ with the metric written in the T-dual coordinates by [8]

$$ds^2 = \frac{-dy_0^2 + dy_1^2 + dr^2}{r^2}. \tag{2}$$

Here we take the static gauge where $(y_0, y_1)$ are regarded as worldsheet directions to write the Nambu–Goto action

$$S = \frac{R^2}{2\pi}\int dy_0\, dy_1 \frac{1}{r^2}\sqrt{D},$$
$$D = 1 + \left(\frac{\partial r}{\partial y_1}\right)^2 - \left(\frac{\partial r}{\partial y_0}\right)^2, \tag{3}$$

from which the equation of motion for $r$ is given by

$$\frac{2\sqrt{D}}{r^3} = \partial_0\left(\frac{\partial_0 r}{r^2\sqrt{D}}\right) - \partial_1\left(\frac{\partial_1 r}{r^2\sqrt{D}}\right). \tag{4}$$

The solution (1) is confirmed to satisfy Eq. (4) with $\sqrt{D} = i$, which implies that the Lagrangian is purely imaginary when it is evaluated on the solution (1). Then the amplitude $\mathcal{A} \sim e^{iS}$ has an exponential suppression factor. The Poincaré coordinates in $AdS_5$ with the boundary $r = 0$,

$$ds^2 = \frac{-dy_0^2 + dy_1^2 + dy_2^2 + dy_3^2 + dr^2}{r^2}, \tag{5}$$

are related to the embedding coordinates $Y_M$ ($M = -1, 0, \ldots, 4$) on which the conformal SO(2, 4) transformation is acting linearly by the following relations:

$$Y^\mu = \frac{y^\mu}{r}, \quad \mu = 0, \ldots, 3,$$
$$Y_{-1} + Y_4 = \frac{1}{r}, \qquad Y_{-1} - Y_4 = \frac{r^2 + y_\mu y^\mu}{r},$$
$$-Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1. \tag{6}$$

The elementary 1-cusp solution (1) is expressed in terms of $Y_M$ as

$$Y_0^2 - Y_{-1}^2 = Y_1^2 - Y_4^2, \qquad Y_2 = Y_3 = 0. \tag{7}$$

Let us make an SO(2, 4) transformation defined by

$$\begin{pmatrix} Y_0 \\ Y_{-1} \end{pmatrix} = P\begin{pmatrix} Y_0' \\ Y_{-1}' \end{pmatrix}, \qquad \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} = Q\begin{pmatrix} Y_1' \\ Y_2' \\ Y_3' \\ Y_4' \end{pmatrix}, \tag{8}$$

with

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \equiv P_1,$$
$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \equiv Q_1. \tag{9}$$

This SO(2) × SO(4) rotation of the elementary 1-cusp solution (7) generates a configuration

$$Y_0' Y_{-1}' = Y_1' Y_4', \tag{10}$$

which is equivalently expressed in terms of the Poincaré coordinates as

$$r = \sqrt{y_0^2 - y_1^2 - \frac{y_0 - y_1}{y_0 + y_1}}, \tag{11}$$

where the prime has been suppressed for convenience. The string configuration (11) is confirmed to satisfy the string equation (4) with $\sqrt{D}$ compactly given by $i/|y_0 + y_1|$. The solution (11) shows that the surface ends on the lines specified by $y_0 = y_1, y_0 = -y_1 \pm 1$ where two cusps are located at $(y_0, y_1) = (\pm 1/2, \pm 1/2)$.

The SO(2, 4) transformations given by $P = 1_2$, $2 \times 2$ unit matrix, $Q = Q_1$ and $P = -i\sigma_2$ that interchanges $Y_0$ and $Y_{-1}$, $Q = Q_1$ produce the following configurations

$$Y_0'^2 - Y_{-1}'^2 = -2Y_1' Y_4', \qquad Y_0'^2 - Y_{-1}'^2 = 2Y_1' Y_4', \tag{12}$$

respectively, which turn out to be

$$r^2 = y_0^2 - (y_1 + 1)^2 \pm 2\sqrt{y_0^2 + (y_1 + 1)^2 - 1}, \tag{13}$$

$$r^2 = y_0^2 - (y_1 - 1)^2 \pm 2\sqrt{y_0^2 + (y_1 - 1)^2 - 1}. \tag{14}$$

In order to show that the latter surface equation obeys the string Eq. (4) we parametrize $r$ as $r = \sqrt{y_0^2 - (y_1 - 1)^2 + 2\sqrt{B}} \equiv \sqrt{A}$ for the plus sign, and $B \equiv y_0^2 + (y_1 - 1)^2 - 1$. In this case $\sqrt{D}$ is again a pure imaginary $\sqrt{D} = i/\sqrt{B}$. The RHS of (4) has also two parts

$$\frac{1}{iA^{3/2}\sqrt{B}}\left[2B + (y_1 - 1)^2 + y_0^2\right] + \frac{3}{iA^{5/2}\sqrt{B}}$$
$$\times\left[(y_1 - 1)^2(1 - \sqrt{B})^2 - y_0^2(1 + \sqrt{B})^2\right], \tag{15}$$

whose second part again becomes proportional to $1/A^{3/2}$ and combines with the first part to yield $2i/A^{3/2}\sqrt{B}$ which is just the left-hand side (LHS) of (4).

For the plus sign of (14) at the boundary of $AdS_3$, $r = 0$, the surface ends on two lines $y_0 = -y_1 + \sqrt{2} + 1$, $y_0 = y_1 - (\sqrt{2} + 1)$ in $y_1 \geqslant 1 + 1/\sqrt{2}$ and two lines $y_0 = -y_1 - (\sqrt{2} - 1)$,