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Generalized Bloch's theorem for viscous metamaterials: Dispersion and effective properties based on frequencies and wavenumbers that are simultaneously complex

Théorème de Bloch généralisé pour les métamatériaux visqueux : dispersion et propriétés effectives fondées sur les fréquences et nombres d'onde simultanément complexes

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A B S T R A C T

It is common for dispersion curves of damped periodic materials to be based on real frequencies as a function of complex wavenumbers or, conversely, real wavenumbers as a function of complex frequencies. The former condition corresponds to harmonic wave motion where a driving frequency is prescribed and where attenuation due to dissipation takes place only in space alongside spatial attenuation due to Bragg scattering. The latter condition, on the other hand, relates to free wave motion admitting attenuation due to energy loss only in time while spatial attenuation due to Bragg scattering also takes place. Here, we develop an algorithm for 1D systems that provides dispersion curves for damped free wave motion based on frequencies and wavenumbers that are permitted to be simultaneously complex. This represents a generalized application of Bloch's theorem and produces a dispersion band structure that fully describes all attenuation mechanisms, in space and in time. The algorithm is applied to a viscously damped mass-in-mass metamaterial exhibiting local resonance. A frequency-dependent effective mass for this damped infinite chain is also obtained.

R É S U M É

Les courbes de dispersion des matériaux périodiques amortis sont habituellement basées soit sur des fréquences réelles en fonction de nombres d'onde complexes, soit sur des nombres d'onde réels en fonction de fréquences complexes. Le premier cas correspond à la propagation d'ondes harmoniques, dont la fréquence d'excitation est imposée, et dont l'atténuation due à la dissipation survient uniquement dans l'espace, en même temps que l'atténuation spatiale due à la diffraction de Bragg. Le second cas concerne la propagation d'ondes libres dont l'atténuation est due à une perte d'énergie dans le temps, en plus de

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l'atténuation spatiale causée par la diffraction de Bragg. Dans cet article, nous développons un algorithme pour des systèmes unidimensionnels afin d'obtenir—pour le mouvement d'ondes libres amorties—les courbes de dispersion fondées sur des fréquences et des nombres d'onde qui sont autorisés à être simultanément complexes. Cette application généralisée du théorème de Bloch fournit une structure de bandes qui décrit pleinement tous les mécanismes d'atténuation, dans l'espace comme dans le temps. L'algorithme est appliqué à un metamatériau à résonance locale (masse incluse dans une masse) amorti de façon visqueuse. Une masse effective dépendant de la fréquence est également obtenue pour cette chaine infinie amortie.

1. Introduction

Phononic materials have been attracting much attention in the materials physics and engineering communities because of their rich scope of acoustic and elastodynamic properties [1–5]. There are two classes of phononic materials: phononic crystals [6,7] and locally resonant acoustic/elastic metamaterials [8]. Phononic crystals resemble atomic-scale crystals in that they consist of repeated units in space. The dispersion curves for elastic wave propagation in a phononic crystal, or a periodic material in general, appear in bands and in some cases band gaps may arise. Bragg scattering is the prime band-gap opening mechanism in a phononic crystal. Metamaterials are usually also periodic, although not by necessity. The prefix *meta-* (meaning "beyond") is associated with the notion that through deliberate design of the internal structure, these materials manifest unusual properties that exceed those of conventional composites and phononic crystals. In the context of wave propagation, local resonances are generally the key feature in metamaterials leading to salient properties such as subwavelength band gaps [9–12], negative effective material properties [13–16], enhanced dissipation [17–19], and thermal conductivity reduction [20], among others.

At present, much of the phenomena of wave propagation in phononic materials is understood from the perspective of conservative linear elasticity. Realizing their full potential, however, requires an account of energy dissipation from damping. Already, metamaterials possessing internal resonating bodies have been shown to demonstrate enhanced dissipation under certain conditions (i.e., beyond which may be attributed to the sum of the individual material constituents) [17–19]. This property is beneficial where enhanced dissipation in a structure is needed but without appearing at the expense of stiffness. There are numerous avenues for the treatment of damping in material or structural models, including those representing phononic materials. A common approach is to consider viscous damping, for which a simple version is known as Rayleigh [21], or proportional, damping—whereby the matrix of damping coefficients is assumed to be proportional to the mass and/or stiffness matrices [22–24]. If the proportionality condition is not met, then the model is described as *generally damped* [25,26]. Experiments are used to determine an appropriate damping model for a given material or structure [27].

Beyond the choice of the damping model, an important consideration is whether the frequency or the wavenumber is selected to be real and, consequently, which is permitted to be complex. There are two classes of problems dealing with damped phononic materials. In one class, the frequencies are assumed *a priori* to be real thus allowing the effects of damping to manifest only in the form of complex wavenumbers. Physically, this represents a medium experiencing wave propagation due to a sustained driving frequency and dissipation taking effect in the form of spatial attenuation only. This approach follows a $\kappa = \kappa(\omega)$ formulation (where κ and ω denote wavenumber and frequency, respectively) resulting from either a linear [28–33] or a quadratic [34–36] eigenvalue problem (EVP). In the other class, the frequencies are permitted to be complex thus allowing dissipation to take effect in the form of temporal attenuation. Physically, this represents a medium admitting free dissipative wave motion, e.g., due to impulse loading. Here, a $\omega = \omega(\kappa)$ formulation leading to a linear EVP is the common route (in some cases with the aid of a state-space transformation); see Refs. [37–41,33].

In the 'driven waves' path, a real frequency is prescribed and the underlying EVP is solved for a corresponding pair of real and imaginary wavenumbers, representing propagation and attenuation constants, respectively. All modes are described by complex wavenumbers due to the dissipation. In the 'free waves' path, on the other hand, a real wavenumber is specified, and complex frequencies emerge as the solution (the real and imaginary parts respectively provide the loss factor and the frequency for each mode). Because of the common association of the driven waves problem to an EVP for which the frequency is the independent variable and, in contrast, the free waves problem to an EVP for which the wavenumber is the independent variable, it is often viewed that the two only available options are: real frequencies and complex wavenumbers versus real wavenumbers and complex frequencies [42–44]. However, if the medium permits spatial attenuation in its undamped state—which is the case for phononic materials within band-gap frequencies—then, in principle, there should be an imaginary wavenumber component (in addition to the real wavenumber component) even when the frequencies are complex. This, in fact, represents a more complete picture of the dispersion curves for damped free wave motion in media that contain inherent mechanisms for spatial attenuation, such as Bragg scattering and local resonance. Since this scenario pertains only to free waves, one expects to see complex frequencies for bands admitting only spatial propagation as well as bands admitting evanescent waves (with the real part of the wavenumber being either zero or π divided by the lattice