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$\mathcal{PT}$-symmetry, indefinite damping and dissipation-induced instabilities

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ABSTRACT

With perfectly balanced gain and loss, dynamical systems with indefinite damping can obey the exact $\mathcal{PT}$-symmetry being marginally stable with a pure imaginary spectrum. At an exceptional point where the symmetry is spontaneously broken, the stability is lost via passing through a non-semi-simple 1 : 1 resonance. In the parameter space of a general dissipative system, marginally stable $\mathcal{PT}$-symmetric ones occupy singularities on the boundary of the asymptotic stability. To observe how the singular surface governs dissipation-induced destabilization of the $\mathcal{PT}$-symmetric system when gain and loss are not matched, an extension of recent experiments with $\mathcal{PT}$-symmetric LRC circuits is proposed.

1. Introduction

The notion of $\mathcal{PT}$-symmetry entered modern physics mainly from the side of quantum mechanics. Parametric families of non-Hermitian Hamiltonians having both parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) symmetry, possess pure real spectrum in some regions of the parameter space, which questions need for the Hermiticity axiom in quantum theory [1–4]. First experimental evidence of $\mathcal{PT}$-symmetry and its violation came, however, from classical optics in media with inhomogeneous in space gain and damping [5,6] and electrodynamics [7].

$\mathcal{PT}$-symmetric equations of two coupled ideal LRC circuits, one with gain and another with loss, have the form

$$\ddot{\mathbf{z}} + \mathbf{D}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = 0, \qquad (1)$$

where dot stands for time differentiation and the real matrix of potential forces is $\mathbf{K} = \mathbf{K}^T > 0$ while the real matrix $\mathbf{D} = \mathbf{D}^T$ of the damping forces is indefinite [7].

For the problem considered in [7], we assume that

$$\mathbf{D} = \mathbf{D}_{\mathcal{PT}} = \begin{pmatrix} -\delta & 0 \\ 0 & \delta \end{pmatrix}, \qquad \mathbf{K} = \mathbf{K}_{\mathcal{PT}} = \begin{pmatrix} k & \kappa \\ \kappa & k \end{pmatrix}, \qquad (2)$$

$\mathbf{z}^T = (z_1, z_2)$, and δ, κ and k are non-negative parameters. Eigenvalues of $\mathbf{D}_{\mathcal{PT}}$ have equal absolute values and differ by sign, indicating perfect gain/loss balance in system (1) with matrices (2). The coordinate change $x_1 = z_1 + iz_2$, $x_2 = x_1^*$, $x_3 = \dot{x}_1$, and $x_4 = \dot{x}_2$, where $i = \sqrt{-1}$ and the asterisk denotes complex conjugation, reduces this system to $i\dot{\mathbf{x}} = \mathbf{H}\mathbf{x}$, where the Hamiltonian

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -ik & \kappa & 0 & i\delta \\ -\kappa & -ik & i\delta & 0 \end{pmatrix} \qquad (3)$$

is $\mathcal{PT}$-symmetric ($\mathbf{PH}^* = \mathbf{HP}$, $\mathbf{P} = \mathrm{diag}(1, -1, -1, 1)$) [8,9].

In real electrical networks, additional losses may result in the indefinite damping matrices that possess both positive and negative eigenvalues with non-equal absolute values. A systematic study of dynamical systems (1) with such a general *indefinite damping*, has been initiated in [10,11] in the context of distributed parameter control theory and population biology [12–14]. In [15–17] gyroscopic stabilization of system (1) was considered, because negative damping produced by the falling dependence of the friction coefficient on the sliding velocity, feeds vibrations in rotating elastic continua in frictional contact, e.g. in the singing wine glass [18–21]. In [22] a gyroscopic $\mathcal{PT}$-symmetric system with indefinite damping was shown to originate in the studies of modulational instability of a traveling wave solution of the nonlinear Schrödinger equation (NLS) [23]. In nonlinear optics, a challenging problem of stability of localized solutions (solitons) is related to the indefinite damping, because stable pulses in dual-core systems frequently exist far from the conditions that provide a perfect matching of gain and loss ($\mathcal{PT}$-symmetry) [24,25]. Recent techniques proposed for the stabilization of the solitons in two coupled perturbed NLSs include introduction of $\mathcal{PT}$-symmetric nonlinear gain and loss [26] which signs can be periodically switched [27,28]. Therefore, indefinite damping is a basic model to study how a localized supply of energy modifies the dissipative structure of a system [14].

In general, the eigenvalues (λ) of system (1), when it is assumed that $\mathbf{z} \sim \exp(\lambda t)$, are complex with positive or negative real parts corresponding either to growing or decaying in time solutions, respectively. *Asymptotic stability* means decay of all modes.

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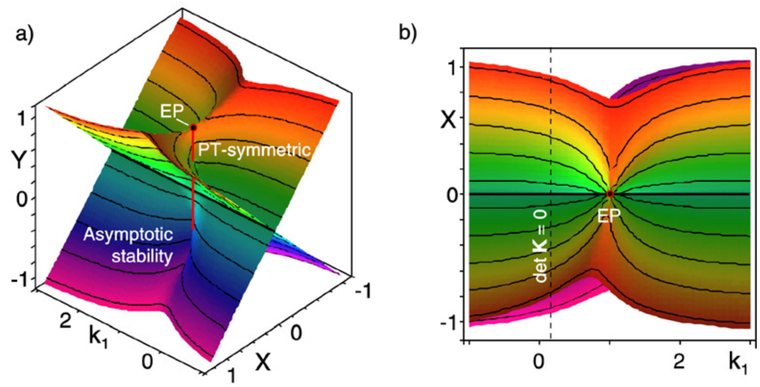

Fig. 1. (a) In the half-space $X > 0$ of the (k_1, X, Y) space, where $X = \delta_1 + \delta_2$ and $Y = \delta_1 - \delta_2$, a part of the singular surface locally equivalent to the Plücker conoid of degree $n = 1$, bounds the domain of asymptotic stability of system (1) with matrices (6) and $\kappa = 0.4$ and $k_2 = 1$; $\mathcal{PT}$-symmetric marginally stable systems occupy the red interval of self-intersection with two exceptional points (EPs) (black dots) at its ends. (b) The top view of the surface. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this Letter.)

A two-dimensional system (1) with $\mathbf{D} = \delta\widetilde{\mathbf{D}}$ is asymptotically stable if and only if $\operatorname{tr}\widetilde{\mathbf{D}} > 0$ and $0 < \delta^2 < \delta_{cr}^2$,

$$\delta_{cr}^2 = \frac{(\operatorname{tr}\mathbf{K}\widetilde{\mathbf{D}} - \sigma_1(\mathbf{K})\operatorname{tr}\widetilde{\mathbf{D}})(\operatorname{tr}\mathbf{K}\widetilde{\mathbf{D}} - \sigma_2(\mathbf{K})\operatorname{tr}\widetilde{\mathbf{D}})}{-\det\widetilde{\mathbf{D}}\operatorname{tr}\widetilde{\mathbf{D}}(\operatorname{tr}\mathbf{K}\widetilde{\mathbf{D}} - \operatorname{tr}\mathbf{K}\operatorname{tr}\widetilde{\mathbf{D}})}, \tag{4}$$

where $\sigma_1(\mathbf{K})$ and $\sigma_2(\mathbf{K})$ are eigenvalues of $\mathbf{K}$ [10,29]. However, when simultaneously $\operatorname{tr}\widetilde{\mathbf{D}} = 0$ and $\operatorname{tr}\mathbf{K}\widetilde{\mathbf{D}} = 0$, the spectrum of the system (1) is *Hamiltonian*, i.e. its eigenvalues are symmetric with respect to the imaginary axis of the complex plane [11]. They are pure imaginary and simple (*marginal stability*) if and only if $\delta^2 < \delta_{\mathcal{PT}}^2$,

$$\delta_{\mathcal{PT}} = \left|\sqrt{\sigma_1(\mathbf{K})} - \sqrt{\sigma_2(\mathbf{K})}\right|(-\det\widetilde{\mathbf{D}})^{-1/2}. \tag{5}$$

How the marginal stability domain of a indefinitely damped $\mathcal{PT}$-symmetric system relates to the domain of asymptotic stability of a nearby dissipative system without this symmetry? The answer is counterintuitive already for the thresholds (4) and (5). Our Letter describes mutual location of the two sets, thus linking the fundamental concepts of modern physics: $\mathcal{PT}$-symmetry [1–4] and dissipation-induced instabilities [30–32].

2. Potential system with indefinite damping

First, we extend the model (1) with matrices (2) by choosing the matrices of damping and potential forces in the form

$$\mathbf{D} = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} k_1 & \kappa \\ \kappa & k_2 \end{pmatrix}, \tag{6}$$

where parameters can take arbitrary positive and negative values. For asymptotic stability it is necessary that $\operatorname{tr}\mathbf{D} > 0$ and $\det\mathbf{K} > 0$ [29].

Introducing the parameters $X = \delta_1 + \delta_2$ and $Y = \delta_1 - \delta_2$, we use the Routh–Hurwitz stability threshold (4) where one should equate the right-hand side to unity and replace the matrix $\widetilde{\mathbf{D}}$ with that given in Eq. (6). The result is a quadratic equation for k_1. Expanding $k_1(X)$ in the vicinity of $X = 0$, yields a linear approximation to the threshold of asymptotic stability in the (k_1, X) plane

$$k_1 = k_2 + \frac{1}{4}\frac{X}{Y}\left[Y^2 \pm \sqrt{\left(Y^2 - {Y_{\mathcal{PT}}^{-}}^2\right)\left(Y^2 - {Y_{\mathcal{PT}}^{+}}^2\right)}\,\right]. \tag{7}$$

$Y_{\mathcal{PT}}^{\pm} = 2(\sqrt{\sigma_2(\mathbf{K})} \pm \sqrt{\sigma_1(\mathbf{K})})$, where $\sigma_1 = k_2 - \kappa$ and $\sigma_2 = k_2 + \kappa$ are eigenvalues of the matrix $\mathbf{K}$ from Eq. (6) in which $k_1 = k_2$ that happens when $X = 0$, i.e. $\delta_1 = -\delta_2$. Therefore, on the line defined by the equations $k_1 = k_2$ and $X = 0$ in the (k_1, X, Y) space, system (1) with the matrices (6) is reduced to the $\mathcal{PT}$-symmetric system with matrices (2) that is marginally stable on the interval $-Y_{\mathcal{PT}}^{-} < Y < Y_{\mathcal{PT}}^{-}$, cf. Eq. (5).

In Fig. 1(a) the vertical red line denotes this interval with $Y_{\mathcal{PT}}^{-} \simeq 0.817$ calculated for $k_2 = 1$ and $\kappa = 0.4$. Along it $\mathcal{PT}$-symmetry is *exact*, i.e. eigenvectors are also $\mathcal{PT}$-symmetric [1–4]. Hence, the spectrum is pure imaginary, see Fig. 2. The ends of the interval are *exceptional points* (EPs) [33] corresponding to the merging of a pair of pure imaginary eigenvalues into a double one with the Jordan block. Passing through these points of the non-semi-simple 1 : 1 resonance with the increase of $|Y|$ is accompanied by the spontaneous breaking of the $\mathcal{PT}$-symmetry of eigenvectors although the system still obeys the symmetry. This causes bifurcation of the double pure imaginary eigenvalues into complex ones with negative and positive real parts and oscillatory instability or *flutter* when ${Y_{\mathcal{PT}}^{-}}^2 < Y^2 < {Y_{\mathcal{PT}}^{+}}^2$, see Fig. 2(a). The bifurcation at $Y^2 = {Y_{\mathcal{PT}}^{+}}^2$ makes all the eigenvalues real of both signs (static instability or *divergence*).

What happens with the stability near the red line in Fig. 1(a)? Fig. 2(b) shows that, e.g. at the fixed $k_1 = 1.2$ and $X = 0.2$, the eigencurves connected at the EPs with $Y = \pm Y_{\mathcal{PT}}^{-}$ in Fig. 2(a), unfold into two non-intersecting loops in the $(\operatorname{Re}\lambda, \operatorname{Im}\lambda, Y)$ space, manifesting an *imperfect merging of modes* [34] owing to gain/loss imbalance.

Now the stability is lost not via the passing through the non-semi-simple 1 : 1 resonance but because of migration of a pair of simple complex-conjugate eigenvalues from the left- to right-hand sides of the complex plane at $|Y| < Y_{\mathcal{PT}}^{-} \simeq 0.817$. For example, tending the parameters to the point (1, 0) in (k_1, X) plane along a ray, specified by the equation $X = k_1 - 1$, we find that the thresholds of asymptotic stability converge to the limiting values of $Y_+ \simeq 0.615 < 0.817$ and $Y_- \simeq -0.531 > -0.817$, see Fig. 2(c, d). The limits vary with the change of the slope of the ray. Therefore, infinitesimal imperfections in the loss/gain balance and in the potential, destroying the $\mathcal{PT}$-symmetry, can significantly decrease the interval of asymptotic stability with respect to the marginal stability interval.

Such a paradoxical finite jump in the instability threshold caused by a tiny variation in the damping distribution, typically occurs in dissipatively perturbed autonomous Hamiltonian or reversible systems [29,35] of structural and contact mechanics [30, 32,34] and hydrodynamics [36–38], as well as in periodic non-autonomous ones [39]. We have just described a similar effect when the marginally stable system is dissipative but obeys $\mathcal{PT}$-symmetry.