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Fractional-order formulation of power-law and exponential distributions

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ABSTRACT

We present a novel approach to data analysis using fractional order calculus. In principle the approach can be applied to any distribution and shows remarkable improvement even if the parameters of a particular distribution have been optimised to achieve the best fit to data. The method is demonstrated for two important distributions that are used in data analysis, namely, power-law and exponential distributions. We show that the approach can allow composite distributions to be constructed for improved accuracy and robustness in the characterisation of data sets.

1. Introduction

Fractional calculus has been around from the time when Leibniz and Newton invented conventional calculus. The word 'fractional' may be a misnomer but has been retained for historical reasons that date back to 1695 when for the first time Leibniz and L'Hopital pondered what the significance of the derivative of order one-half might mean. A more appropriate definition would be 'generalised' calculus instead of 'fractional' calculus. Conventional calculus deals with integer order differentiation and integration while fractional calculus deals with differentiation and integration of arbitrary order that includes real numbers and complex numbers. Indeed, conventional calculus is a special limit of fractional calculus when the order of differentiation and/or integration is an integer. Fractional order calculus had been sidelined as a mathematical tool by researchers mainly because of its increased complexity compared to conventional calculus. Another reason is that until recently is was not very clear as to what the physical significance of non-integer calculus is. For example, consider the position of a particle $s(t)$ as a function of time for an initial velocity u and acceleration a:

$$s(t) = ut + \frac{1}{2}at^2 \tag{1}$$

The first order derivative with respect to time gives the velocity while the second order derivative gives the acceleration:

$$v(t) \equiv \frac{ds(t)}{dt} = u + at \quad \text{and}$$

$$a(t) \equiv \frac{d^2 s(t)}{dt^2} = a \tag{2}$$

respectively. These equations are well known in classical mechanics and they are related via the use of conventional calculus. An interesting question arises: what is the physical interpretation of the half derivative of $s(t)$: $d^{0.5}s(t)/dt^{0.5}$? Does the half derivative of (1) describe the particle's position, velocity or a mixture of both? Likewise does the half derivative of the particle's velocity describe its velocity, acceleration or a mixture of both?

Integer order derivatives describe how a function varies at a point while the non-integer derivative describes the behaviour of the function at a point but also at surrounding points or 'regions'. If no derivative is taken then the particle's state is described by the position (1). The first order derivative gives 'precise' information about the particle's velocity. However by taking the half-derivative of the position of the particle $d^{0.5}s(t)/dt^{0.5}$ we gain knowledge of both the position and velocity, a kind of mixture or fuzziness between the two states or roughly 50% knowledge about its position and 50% knowledge about its velocity. In the same way, the same conclusions can be drawn for the case of velocity and acceleration in (2). This is demonstrated in Fig. 1 which shows the differential variation for integer and non-integer order α. For $\alpha = 0$, the distance covered by the particle as a function of time is shown for initial velocity $u = 10$ ms^{-1} and acceleration $a = 4$ ms^{-2}. The integer order derivative $\alpha = 1$ gives the velocity, i.e., $v(t) = s^{(1)}(t)$ while the second (integer) order $\alpha = 2$ represents the particle's acceleration, i.e., $a(t) = s^{(2)}(t)$. The non-integer order $\alpha = 1/2$ describes the particle motion as a 'mixture' made up from information on its position and its velocity roughly in equal measure. In a similar way $\alpha = 3/2$ represents the particle in a state that describes both its velocity and acceleration. It should be evident that any order α

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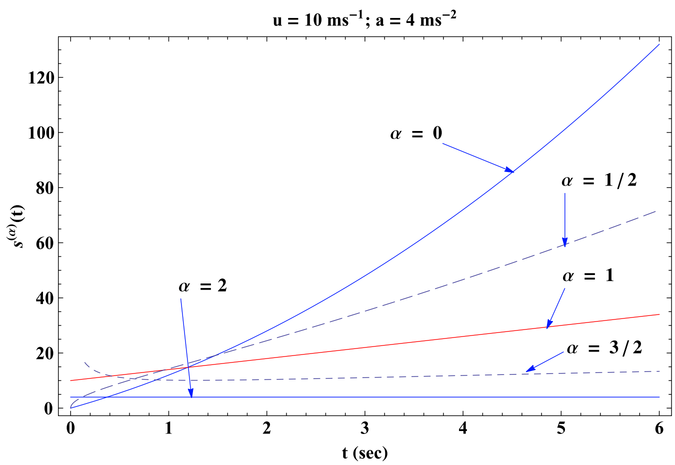

Fig. 1. The variation of the particle position $s^{(\alpha)}(t)$ for fractional and integer order derivatives α.

can be selected to describe the asymptotic behaviour of the curves representing the motion of the particle. This aspect is generalisable and can be applied to any type of function or curve. Thus, in general terms, if the 'velocity' curve represents an arbitrary optimal fit to a particular data set for example, it is possible to change the characteristics of such a curve asymptotically using the fractional order α. A choice of $\alpha = 0.95$ will represent a curve that fits the optimal curve (the velocity curve here) from above while $\alpha = 1.1$ will fit the optimal curve from below. As $\alpha \to 1$, the first order *integer* derivative, the fractional order curve and the optimal curve are equal. It is worth pointing out that the same principle holds if we were to use integration (fractional and integer) instead.

We extend these ideas and develop a new approach that applies to any type of distribution in principle while improving the ability of a particular distribution to mathematically fit a given data set. The technique also allows the straightforward construction of composite distributions by 'merging' the fractional version of unrelated conventional distributions. In order to demonstrate the concept further we have used fractional calculus to derive fractional-order power-law and exponential distributions for fitting experimental data. We show that the idea of using fractional-order distributions allows the fitting of data more accurately even after the parameters in these distributions have already been optimised using the maximum likelihood method [1–3] and no further accuracy is possible under conventional analysis.

2. Fractional-order power and exponential distributions

In conventional calculus differentiation (or integration) is performed to integer order, e.g. $\frac{d^n}{dx^n} f(x)$ where $n \in N$. In the fractional calculus approach we examine here, the fractional order of differentiation or integration α can be real or complex, i.e., $\alpha \in R$, $\alpha \in C$. Fractional order integration can be performed using [4,5]:

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(x)(t-x)^{\alpha-1} dx \tag{3} $$

The fractional differentiation of a function $f(x)$ is obtained by using the so-called Riemann–Liouville integro-differential equation [4,5]:

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx \tag{4} $$

where a and t are the integration limits. Notice that, contrary to conventional calculus, in order to obtain the fractional order

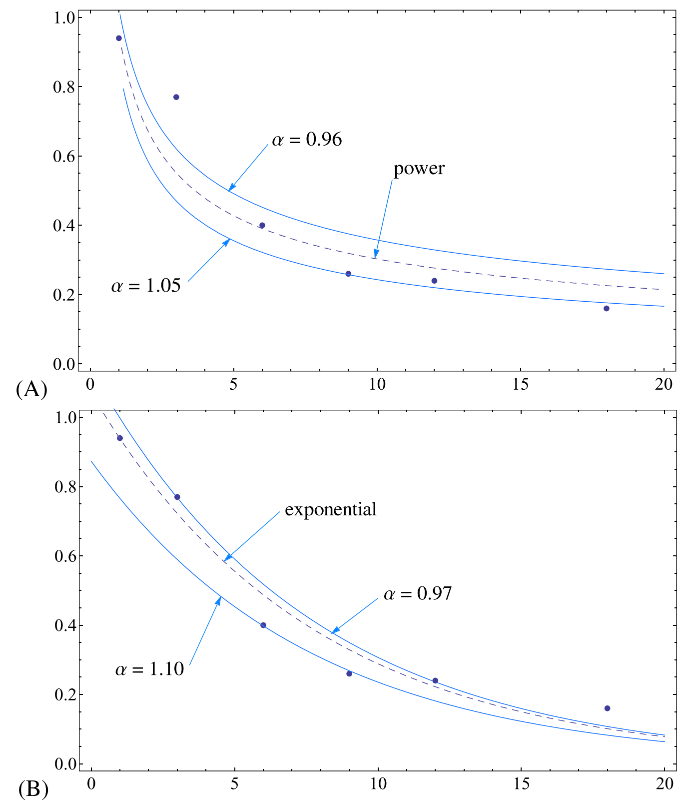

Fig. 2. Subfigure (A): Distributions for power law (6) and fractional-power law (8) for $\alpha = 0.96$ and $\alpha = 1.05$. As α changes in the interval $[0.96, 1.05]$ the asymptotic behaviour of the fractional distribution changes accordingly. Subfigure (B): Distributions for exponential law (9) and fractional-exponential law (11) for $\alpha = 0.97$ and $\alpha = 1.10$. Once again as α changes in the interval $[0.97, 1.10]$ the asymptotic behaviour of the fractional distribution also changes. For $\alpha = 1$, the power-exponential distributions and fractional power-exponential distributions are exact. Dots represent data from [6].

derivative of a function one must use integration as can be seen in (4). The integer order derivative is chosen such that $n = \lceil \alpha \rceil$, i.e., the ceiling function of α. For example if $\alpha = 1/2$ then $n = 1$. Both expressions (3) and (4) make use of the gamma-function,

$$ \Gamma(t) = \int_0^\infty e^{-x} x^{t-1} dx \tag{5} $$

with $Re(z) > 0$. Furthermore we note the properties $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(n) = (n-1)!$ The operator ${}_aD_t^{-\alpha} f(t)$ represents fractional integration of the function $f(t)$. This can also be represented by the notation $d^{-\alpha} f(t)/dt^{-\alpha}$. Changing the sign of α allows us to obtain the fractional differentiation ${}_aD_t^{\alpha} f(t)$ which can also be represented by the notation $d^{\alpha} f(t)/dt^{\alpha}$.

Use of the Riemann–Liouville equation is made, i.e. (4), to derive fractional distributions in order to fit the experimental data found in [6,7] while comparing to the optimised conventional power and exponential distributions. The data pertains to neurological experiments and is: $[x_i, y_i] = [(1, 0.94), (3, 0.77), (6, 0.40), (9, 0.26), (12, 0.24), (18, 0.16)]$ where the y-values represent memory retention vs time in seconds (x-axis). All plot axes presented in this Letter concern memory retention vs time but labelling has been omitted for brevity. The conventional power distribution we will compare with has the form:

$$ p_{p.} = \omega_1 x^{-\omega_2} \tag{6} $$

The parameters ω_1 and ω_2 are optimised using the maximum likelihood estimator (MLE) approach and we will be using the same