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From fractional exclusion statistics back to Bose and Fermi distributions

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ABSTRACT

Fractional exclusion statistics (FES) is a generalization of the Bose and Fermi statistics. Typically, systems of interacting particles are described as ideal FES systems and the properties of the FES systems are calculated from the properties of the interacting systems. In this Letter I reverse the process and I show that a FES system may be described in general as a gas of quasiparticles which obey Bose or Fermi distributions; the energies of the newly defined quasiparticles are calculated starting from the FES equations for the equilibrium particle distribution. In the end I use a system in the effective mass approximation as an example to show how the procedure works.

1. Introduction

The concept of fractional exclusion statistics (FES) was introduced by Haldane in Ref. [1] and the statistical mechanics of FES systems was formulated by several authors, employing different methods [2–8]. The FES have been applied to the description of several types of interacting particle systems [1–7,9–15] and the properties of FES systems in different numbers of dimensions and trapping potentials have been studied [16–31]. A general method for the description of Fermi liquid type of systems as ideal FES gases have been proposed in Refs. [32,33] and the time evolution of FES systems in non-equilibrium have been studied by Monte Carlo simulations in Ref. [34].

Typically, for systems described in the mean field approximation, like the ones analyzed in Refs. [13–15,32,33,35], one employs Landau's Fermi liquid formalism in which quasiparticle energies are defined in such a way that the equilibrium populations are Bose or Fermi distributions (depending on whether we have bosons or fermions in the system) over the quasiparticle energies. The quasiparticle system is neither ideal, nor it is thermodynamically equivalent with the initial interacting particle system, but if one properly redefines the quasiparticle energies, one may transform it into an ideal gas, thermodynamically equivalent with the original system [14]. The ideal gas thus obtained obeys FES.

The connection between FES and Fermi liquid theory was investigated in more detail in Ref. [35] where it was shown that a Bose or a Fermi system in the Thomas–Fermi approximation with long-range particle–particle interaction may be described as an ideal FES system with a different type of quasiparticle energies. The populations of the Bose or Fermi quasiparticle states are identical to the populations of the quasiparticle states defined in the FES formalism. Here I reverse the process and I describe a system with statistical interaction, i.e. a FES system, as a Bose or Fermi gas by redefining the quasiparticle energies. The quasiparticle energies in the Bose or Fermi systems are determined from the FES equilibrium distribution equations.

The Letter is organized as follows. First I present FES and the equations that give the equilibrium particle populations. In these equations I introduce the ansatz for Bose or Fermi distributions and I obtain a set of equations for the quasiparticle energies. I particularize these equations for a mean-field gas with a density dependent effective mass to show how the formalism works. In the end I give the conclusions.

2. Particle distribution in FES

For the calculation of the particle distribution in FES one can adopt three equivalent descriptions: bosonic, fermionic [14,36,37] and the standard FES description [4].

2.1. The bosonic description

We have a general FES system of species (G_i, N_i, ϵ_i), where G_i is the number of states, N_i is the number of particles, and ϵ_i is the single-particle energy in the species. The FES parameters, $\alpha_{ij}^{(-)}$, describe the change of the number of states, δG_i, at a change of the number of particles, δN_j: $\delta G_i = -\alpha_{ij}^{(-)} \delta N_j$ for any i and j. The grandcanonical partition function is

$$\mathcal{Z}^{(-)} = \sum_{\{N_i\}} \mathcal{Z}^{(-)}_{\{(G_i,N_i)\}} \tag{1}$$

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and if the particles are bosons the partial partition function, $\mathcal{Z}^{(-)}_{\{(G_i,N_i)\}}$, is [36,37]

$$\mathcal{Z}^{(-)}_{\{(G_i,N_i)\}} = \prod_j \frac{(G_j+N_j-1)!}{N_j!(G_j-1)!} e^{-\beta(\epsilon_j-\mu)N_j}. \quad (2)$$

If we maximize $\mathcal{Z}^{(-)}_{\{(G_i,N_i)\}}$ subject to the variation of the species populations we obtain the equations for the equilibrium particle distribution,

$$0 = \frac{\partial \log \mathcal{Z}^{(-)}_{\{(G_i,N_i)\}}}{\partial N_k} = \ln \frac{1+n_k^{(-)}}{n_k^{(-)}} - \sum_i \alpha_{ik}^{(-)} \ln\bigl(1+n_i^{(-)}\bigr) - \beta(\epsilon_k-\mu) \quad (3)$$

where $n_i^{(-)} \equiv N_i/G_i$ [36].

2.2. The fermionic description

Another way to look at the same problem is to assume that while G_i is the number of available states in the species i, the actual number of states is $T_i \equiv G_i + N_i$ – like in the situation when there are N_i fermions on T_i states. In such a case we define the FES parameters, $\alpha_{ij}^{(+)}$, so that $\delta T_i = -\alpha_{ij}^{(+)}\delta N_j$. The partial partition function is

$$\mathcal{Z}^{(+)}_{\{(T_i,N_i)\}} = \prod_i \frac{T_i!}{N_i!(T_i-N_i)!} e^{-\beta(\epsilon_i-\mu)N_i} \quad (4)$$

and the maximization with respect to N_k gives

$$\ln \frac{1-n_k^{(+)}}{n_k^{(+)}} + \sum_i \alpha_{ik}^{(+)} \ln\bigl(1-n_i^{(+)}\bigr) - \beta(\epsilon_k-\mu) = 0, \quad (5)$$

where $n_i^{(+)} = N_i/T_i$ [36].

The fermionic description is more appropriate for FES in Fermi systems and changes into the bosonic description by the redefinitions $\alpha_{ij}^{(-)} \equiv \delta_{ij} + \alpha_{ij}^{(+)}$, $G_i \equiv T_i - N_i$ and $n_i^{(+)} = n_i^{(-)}/(1-n_i^{(-)})$.

2.3. The general FES description

The bosonic and fermionic descriptions presented above are equivalent to the usual description of FES systems, originally proposed by Wu [4]. First, the equivalence between the bosonic formulation and Wu's formulation may be observed if we define the number of states in the species i when "no particles are in the system" ($N_i = 0$ for any i): $G_i^{(0)} \equiv G_i + \sum_j \alpha_{ij}^{(-)} N_j$ – actually $G_i^{(0)}$ is a linear extrapolation of G_i, assuming $\alpha_{ij}^{(-)}$'s independent of N_i's. Writing the partial partition function (2) in terms of G_i^0,

$$\mathcal{Z}^{(-)}_{\{(G_i,N_i)\}} = \prod_j \frac{(G_j^{(0)} - \sum \alpha_{jk}^{(-)} N_k + N_j - 1)!}{N_j!(G_j^{(0)} - \sum \alpha_{jk}^{(-)} N_k - 1)!} e^{-\beta(\epsilon_j-\mu)N_j}, \quad (6a)$$

and defining $n_i^{(0)} \equiv N_i/G_i^{(0)} = n_i^{(-)}/[1 + \sum_j(\alpha_{ij}^{(-)} n_j^{(-)} G_j/G_i)]$, by the maximization procedure we obtain the system of equations,

$$(1+w_i)\prod_j\left(\frac{w_j}{1+w_j}\right)^{\alpha_{ji}^{(-)}} = e^{(\epsilon_i-\mu)/k_BT}, \quad (6b)$$

$$\sum_j \bigl(\delta_{ij} w_j + \alpha_{ij} G_j^{(0)}/G_i^{(0)}\bigr) n_j^{(0)} = 1. \quad (6c)$$

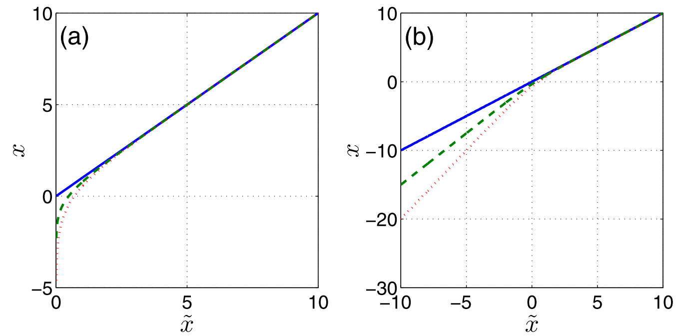

Fig. 1. (Color online.) The variation of x with respect to $\tilde{x}$ for $\alpha_{ij}^{(\pm)} = \alpha\delta_{ij}$ and $\alpha = 0, 0.5, 1$, from up to down (solid-blue, dashed-green and dotted-red lines, respectively) for bosons (a) and fermions (b).

Eq. (6b) should be solved to obtain Wu's auxiliary functions, w_i, which then may be plugged into Eq. (6c) to calculate the equilibrium populations, $n_j^{(0)}$. From Eqs. (3) and (6b) we observe that $w_i \equiv 1/n_i^{(-)}$ [37].

Second, the transformation from the fermionic description to Wu's description is obtained by using $\alpha_{ij}^{(-)} \equiv \delta_{ij} + \alpha_{ij}^{(+)}$, $G_i \equiv T_i - N_i$ and defining $G_i^{(0)} \equiv G_i + \sum_j \alpha_{ij}^{(-)} N_j = T_i + \sum_j \alpha_{ij}^{(+)} N_j$ in terms of the new parameters $\alpha_{ij}^{(-)}$. The equilibrium populations are given again by Eqs. (6). In this case we observe that $w_i = 1/n_i^{(-)} = (T_i - N_i)/N_i = (1-n_i^{(+)})/n_i^{(+)}$, where $1-n_i^{(+)}$ is the holes population.

3. The FES distribution written as a Bose or a Fermi distribution

We look for solutions of the form

$$n_i^{(\pm)} = \frac{1}{e^{\beta(\tilde{\epsilon}_i-\mu)} \pm 1}, \quad (7)$$

for Eqs. (3) and (5) – in what follows we shall always use the upper signs for fermions and the lower signs for bosons. Plugging (7) into Eqs. (3) and (5) we obtain a self-consistent set of equations for the quasiparticle energies,

$$\tilde{\epsilon}_k = \epsilon_k \mp k_BT \sum_i \alpha_{ik}^{(\pm)} \ln\bigl[1 \mp n_i^{(\pm)}\bigr] = \epsilon_k \pm k_BT \sum_i \alpha_{ik}^{(\pm)} \ln\bigl[1 \pm e^{-\beta(\tilde{\epsilon}_i-\mu)}\bigr]. \quad (8)$$

For bosons in the standard FES formulation, the ansatz (7) leads to

$$w_i^{(-)} = e^{\beta(\tilde{\epsilon}_i-\mu)} - 1, \quad (9a)$$

while in the case of fermions we have

$$w_i^{(+)} = e^{\beta(\tilde{\epsilon}_i-\mu)}. \quad (9b)$$

The expression (9b) was obtained in interacting fermionic systems described in the thermodynamic Bethe ansatz by Bernard and Wu [9]. We notice that the ansatz (7) is applicable neither to $n^{(0)}$, nor to w_i, but only to $n_i^{(\pm)}$.

In many situations, $\alpha_{ij}^{(\pm)} \equiv \alpha\delta_{ij}$ (e.g. [6,13,19–21,38]). In such a case

$$\tilde{\epsilon}_k = \epsilon_k \pm \alpha k_BT \ln\bigl[1 \pm e^{-\beta(\tilde{\epsilon}_k-\mu)}\bigr] \quad (10a)$$

$$= \frac{\epsilon_k}{1\pm\alpha} \pm \frac{\alpha k_BT}{1\pm\alpha}\bigl[\mu - \ln n^{(\pm)}(\epsilon_k)\bigr] \quad (10b)$$

or