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Computational approaches to aspect-ratio-dependent upper bounds and heat flux in porous medium convection

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ABSTRACT

Direct numerical simulation (DNS) has shown that Rayleigh–Bénard convection in a fluid-saturated porous medium self-organizes into narrowly spaced plumes at (ostensibly) asymptotically high values of the Rayleigh number *Ra*. In this Letter a combination of DNS and upper bound theory is used to investigate the dependence of the Nusselt number *Nu* on the domain aspect ratio *L* at large *Ra*. A novel algorithm is introduced to solve the optimization problems arising from the upper bound analysis, allowing for the best available bounds to be extended up to $Ra \approx 2.65 \times 10^4$. The dependence of the bounds on $L(Ra)$ is explored and a "minimal flow unit" is identified.

1. Introduction

Thermal convection and other buoyancy-driven flows are fundamental processes in a variety of natural systems and technological applications. In particular, free thermosolutal convection in fluid-saturated porous media is a key environmental process that impacts, e.g., oil recovery, geothermal energy extraction, and groundwater flow [1,2]. Indeed, there has been significant renewed interest in dissolution-driven convection in porous layers owing to the potential impact of this process on carbon dioxide storage in terrestrial aquifers. In addition to these geoscientific applications, porous medium convection also plays an important role in fibrous insulation, the design of compact heat exchangers, and other engineering applications. Moreover, as a paradigm for forced–dissipative infinite-dimensional nonlinear dynamical systems, buoyancy-driven convection in porous media displays much (though not all) of the rich dynamics of Rayleigh–Bénard convection in a pure fluid layer, including a hierarchy of instabilities and bifurcations, pattern formation, and spatiotemporally chaotic dynamics (if not "true" fluid dynamical turbulence).

In this study we explore the dynamics of porous medium convection using the two-dimensional (2D) Darcy–Oberbeck–Boussinesq equations in the infinite Darcy–Prandtl number limit. This mathematical model combines Darcy's law for incompressible flow in a fluid-saturated porous medium and buoyancy forces incorporated through the Boussinesq approximation together with a time-dependent advection–diffusion equation for the temperature field (see Section 2). The sole nonlinearity arises from temperature advection, and consequently this system is somewhat simpler than the full Oberbeck–Boussinesq equations. One crucial phenomenological distinction between Rayleigh–Bénard convection of a pure fluid and that occurring in a fluid-saturated porous layer is the observed mean spacing between adjacent rising and falling thermal plumes, which decreases with increasing Rayleigh number *Ra* in porous medium but not in classical Rayleigh–Bénard convection [3–5]. Indeed, for porous medium convection, linear stability analysis of the purely conducting state reveals that the horizontal wavelength of the smallest unstable mode scales as $Ra^{-1/2}$ while that of the fastest-growing disturbance decreases as $Ra^{-1/4}$. Remarkably, recent direct numerical simulation (DNS) by Hewitt et al. [6] not only shows that the mean inter-plume spacing decreases markedly with *Ra*, scaling approximately as $Ra^{-2/5}$, but also that the flow actually becomes more organized in the interior (i.e. away from

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the upper and lower thermal boundary layers) as Ra is increased. A complementary numerical and matched asymptotic investigation of steady cellular solutions in porous medium convection [7] also confirms that those with the highest heat transport decrease in lateral scale – specifically as $Ra^{-1/2}$, just slightly larger than the smallest horizontal scale capable of sustaining convection. Thus all these investigations confirm the trend toward compression of horizontal scales, but the exact Ra scaling and physical mechanism controlling the inter-plume spacing in high-Ra porous medium convection remain important open questions. Hewitt et al. [8] recently proposed one explanation for the plume spacing based on a Floquet stability analysis of a strictly columnar vertical exchange flow (i.e., in the absence of vertical boundaries).

Motivated by these recent studies we address two related but distinct questions in this Letter. First, is there a smallest domain aspect ratio $L(Ra)$ above which the Nusselt number Nu, the volume- and time-averaged heat flux normalized by the conduction value, becomes independent of L? This question is analogous to that of determining the "minimal flow unit" in wall-bounded shear flow turbulence, the smallest physical domain with horizontally periodic boundary conditions in which (low-dimensional) turbulence can sustain itself. We address this question empirically via highly-resolved DNS utilizing a Fourier–Chebyshev pseudospectral algorithm. The second question is, how do optimal upper bounds on $Nu(Ra)$ obtained from the rigorous "background" (sometimes called Constantin–Doering–Hopf) variational formalism depend on L? Upper bounds for statistically stationary convection in a porous layer were first obtained by Busse [9] and Busse and Joseph [10] using a methodology developed by Howard for Rayleigh–Bénard convection in a pure fluid layer [11]. Twenty years later, the background method [12–15] was used to produce rigorous upper bounds on energy dissipation in shear flows and on heat transport in convection problems without any statistical hypotheses, scaling assumptions, or closure approximations. This approach is based on Hopf's method for producing *a priori* estimates for solutions of the Navier–Stokes equations with inhomogeneous boundary conditions [16]. The basic idea is to decompose the dynamical field into a time-independent background component carrying the inhomogeneous boundary conditions plus a nonlinear fluctuation satisfying homogeneous boundary conditions. It is worth emphasizing that the background field (or, in the case of Rayleigh–Bénard convection, background temperature profile) is neither a steady solution of the governing equations nor a horizontal/long-time mean. Ensuring that appropriate test backgrounds satisfy a certain spectral constraint, which effectively is an energy stability condition, produces rigorous upper bounds on global transport properties of the flow whether it is laminar or turbulent.

Following the analysis in Doering and Constantin [17] and Otero et al. [18], Wen et al. [19] obtained the optimal upper bounds on Nu in porous medium convection for $Ra \leqslant 2102$ by solving the full background variational problem using a standard numerical optimization package. However, for Rayleigh numbers greater than a few thousand this scheme is computationally expensive and insufficiently robust producing, e.g., numerically-induced small-scale oscillations in the associated upper bound eigenfunctions. In this study we propose a new strategy for solving the full background problem efficiently and accurately, and thereby extend the best available bounds on the heat transport in porous medium convection up to $Ra \approx 2.65 \times 10^4$. This new computational approach should be widely applicable to other systems to which the background formalism can be applied (e.g. shear flows [20]). The numerical algorithm is also crucial for the implementation of a novel, fully *a priori* reduced-order modeling strategy proposed by Chini et al. [21], which exploits a spectral expansion in the upper bound functional basis. In Section 2 we outline this new numerical scheme after first recording the mathematical problem formulation. Both DNS and upper bound computations are described and compared in Section 3, and our conclusions are presented in Section 4.

2. Problem formation and computational methodology

We investigate the heat transport in porous medium convection in a 2D domain $(x, z) \in [0, L] \times [0, 1]$ using the dimensionless Darcy–Oberbeck–Boussinesq equations in the infinite Darcy–Prandtl number limit [2]:

$$\partial_t T + \mathbf{u} \cdot \nabla T = \Delta T, \qquad \nabla \cdot \mathbf{u} = 0,$$
$$\mathbf{u} + \nabla P = Ra\, T \mathbf{e}_z \ \left(\Rightarrow \Delta w = Ra\, \partial_x^2 T\right), \tag{1}$$

where $\mathbf{u}(x, z, t) = u(x, z, t)\mathbf{e}_x + w(x, z, t)\mathbf{e}_z$, $P(x, z, t)$, and $T(x, z, t)$ are, respectively, the velocity, pressure, and temperature fields satisfying boundary conditions

$$T|_{z=0} = 1, \qquad T|_{z=1} = 0, \qquad w|_{z=0,1} = 0 \tag{2}$$

and L-periodic in x. Two control parameters govern the dynamics of this system: the domain aspect ratio L and the Rayleigh number $Ra = \alpha g (T_{bot} - T_{top}) K H / (\nu\kappa)$, representing the ratio of driving to damping forces, where α is the thermal expansion coefficient, g is the gravitational acceleration, $T_{bot} - T_{top}$ is the temperature drop across the layer, K is the Darcy permeability coefficient, H is the layer depth, ν is the kinematic viscosity, and κ is the thermal diffusivity. A primary quantity of interest in convection is the Nusselt number Nu, the ratio of the heat transport in the presence of convective motion to the conductive heat transport when $\mathbf{u} = \mathbf{0}$. In terms of solutions to (1), $Nu = 1 + \langle wT \rangle = \langle |\nabla T|^2 \rangle$, where $\langle \cdot \rangle$ denotes a space–time average. We compute, estimate and bound Nu using a combination of DNS and generalized energy stability and upper bound theory, thereby providing the first systematic exploration of the dependence of Nu on L at large Ra. The DNS is performed using a standard Fourier–Chebyshev pseudospectral algorithm with semi-implicit time stepping, while the upper bound computations use a novel two-step algorithm, described below.

In the background analysis, the temperature $T(x, z, t)$ is decomposed into a steady background field $\tau(z)$ plus an arbitrarily large fluctuation $\theta(x, z, t)$. That is,

$$T(x, z, t) = \tau(z) + \theta(x, z, t), \tag{3}$$

where $\tau(0) = 1$, $\tau(1) = 0$, and $\theta|_{z=0,1} = 0$. Doering and Constantin [17] and Otero et al. [18] show that for any $\tau(z)$, $a \in (0, 1)$ and $Ra \geqslant 4\pi^2$,

$$Nu \leqslant 1 + \frac{nu - 1}{4a(1 - a)}, \tag{4}$$

where $nu = \int_0^1 \tau'(z)^2\, dz$, if and only if the "spectral constraint"

$$0 \leqslant \min_{\vartheta} \left\{ \frac{1}{L} \int_0^L dx \int_0^1 dz \left[a|\nabla\vartheta|^2 + \tau' W \vartheta \right] \right\} \tag{5}$$

holds for all non-trivial test functions $\vartheta(x, z)$ satisfying L-periodic boundary conditions in x and homogeneous boundary conditions in z, where $W(x, z)$ solves $\Delta W = Ra\partial_x^2\vartheta$ with L-periodic boundary conditions in x and homogeneous boundary conditions in z as well. For later reference we note that for $a = 1$ the spectral constraint is tantamount to enforcing energy stability about the background profile $\tau(z)$ *as if* it were a steady conduction solution of the Darcy–Oberbeck–Boussinesq equations with suitable sources and sinks.

Here we follow the approach used in Wen et al. [19] to optimize the upper bounds over the "balance parameter" a. First we