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Synchronization of Kuramoto model in a high-dimensional linear space

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ABSTRACT

In this Letter, Kuramoto model in a high-dimensional linear space is investigated. Some results on the equilibria and synchronization of the classical Kuramoto model are generalized to the high-dimensional Kuramoto model. It is proved that, if the interconnection graph is connected and all the initial states lie in a half part of the state space, the synchronization can be achieved. Finally, numerical simulations are given to validate the obtained theoretical results.

1. Introduction and preliminaries

Kuramoto model was first proposed by Y. Kuramoto in 1975 [1], which is a mathematical model describing the synchronization behavior of a group of coupled oscillators. This model has wide background in many fields such as biology, engineering and physics [2–11]. The classic Kuramoto model describes the dynamics of a set of coupled phase oscillators. Kuramoto [1] found that there is a certain value of the coupling strength, above which the frequencies of the oscillators can tend to a same value. After that, Kuramoto model was widely investigated in the physics community. In the recent years, Kuramoto model was also paid great attention by researchers in the control community [12–17], since its dynamical properties are mathematically related to coordination and flocking of nonlinear multi-agent systems. Multi-agent systems have been researched for many years including single-integrator, double integrator, the general linear and nonlinear multi-agent systems [18–23]. Kuramoto model can be regarded as a special nonlinear multi-agent system. In [12], the Kuramoto model with an arbitrary interconnection topology is considered. Based on spectral graph theory and control theory, the synchronization is obtained with a critical value of the coupling strength. Further extension can be found in [13]. In our recent paper [16], the Kuramoto model with the a proximity graph is investigated. Under a limitation on the initial states and the assumption of initial connectedness, the synchronization is achieved. In [14,15], properties of equilibria of the Kuramoto model with identical frequencies are investigated and almost global synchronization is proved for some special interconnection graphs. Kuramoto model actually describes a dynamical behavior on the unit circle. For the case of the natural frequencies being zero, a high-dimensional Kuramoto model on the high-dimensional unit sphere is proposed in [17]. For the all-to-all coupling graph, the stability of the equilibria is analyzed and almost global synchronization is achieved. However, the general high-dimensional Kuramoto model with distinct natural frequencies is not established yet. Moreover, for the general coupling graphs, there are no much results reported on the dynamical properties of the high-dimensional Kuramoto model. Many results on the classical Kuramoto model have not been generalized to the high-dimensional case.

In this Letter, we first propose the more general forms of Kuramoto model with distinct natural frequencies in high-dimensional linear spaces. Next, some properties of the equilibria of the high-dimensional Kuramoto model are obtained. After that, under some limited initial conditions, the synchronization is achieved. Finally, simulations are made to validate the obtained theoretical results.

Consider the multi-agent system with interconnection network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, which is a digraph with the set of nodes $\mathcal{V} = \{1, 2, \dots, m\}$, set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and a nonnegative adjacency matrix $\mathcal{A} = (a_{ij}) \in \{0, \ 1\}$. An edge of $\mathcal{G}$ is denoted by (i, j), which means node j can receive the state information from i. Adjacency matrix $\mathcal{A}$ is defined such that $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$, while $a_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. We denote the set of neighbors of node i by $N_i = \{i_1, \ i_2, \dots, i_{n_i}\}$, where n_i is the number of neighbors of i. The Laplacian matrix of the digraph is defined as $L = (l_{ij})$, where

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$l_{ii} = \sum_{j=1, j\neq i}^{m} a_{ij}$ and $l_{ij} = -a_{ij}$ $(i \neq j)$. Let $\mathbf{1}_n$ denote the $n \times 1$ column vector of all ones. Let I_n denote the $n \times n$ identity matrix. It is obvious that $L\mathbf{1}_n = 0$. In the whole Letter, we denote by $\mathbf{R}$ the real number field.

Suppose each node of the graph is a dynamic agent with dynamics $\dot{x}_i = f_i(x_i, x_{i_1}, x_{i_2}, \dots, x_{i_{n_i}})$, where $x_i \in \mathbf{R}^n$ is the state of the ith agent and $i = 1, 2, \dots, m$.

The *synchronization problem* is finding conditions to guarantee $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$ for all $i, j = 1, 2, \dots, m$.

2. Equilibria of generalized forms of Kuramoto model

Firstly, we give a nonlinear multi-agent system with dynamics as follows:

$$\begin{aligned}\dot{r}_i &= W_i r_i + k\sum_{j\in N_i}\left(r_j - \frac{r_j^{\mathrm{T}} r_i}{r_i^{\mathrm{T}} r_i} r_i\right)\\ &= W_i r_i + k\left(I_n - \frac{r_i r_i^{\mathrm{T}}}{r_i^{\mathrm{T}} r_i}\right)\sum_{j=1}^{m} a_{ij} r_j, \end{aligned} \tag{1}$$

where $r_i \in \mathbf{R}^n$ is the state of agent i, $\mathcal{A} = (a_{ij}) \in \mathbf{R}^{m\times m}$ the adjacency matrix, N_i is the neighborhood of i and $i = 1, 2, \dots, m$.

Proposition 1. *Consider the nonlinear multi-agent system (1). If W_i is a skew symmetric matrix, i.e. $W_i^{\mathrm{T}} = -W_i$, then for every $i = 1, 2, \dots, m$ the value of $\|r_i(t)\|$ is constant, that is, $\|r_i(t)\| = \|r_i(0)\|$ for any $t \geqslant 0$.*

Proof. Let $V_i(t) = r_i^{\mathrm{T}}(t) r_i(t)/2 = \|r_i(t)\|^2/2$. Then

$$\dot{V}_i = r_i^{\mathrm{T}} W_i r_i + k r_i^{\mathrm{T}}\left(I_n - \frac{r_i r_i^{\mathrm{T}}}{r_i^{\mathrm{T}} r_i}\right)\sum_{j=1}^{m} a_{ij} r_j. \tag{2}$$

Since W_i is skew symmetric, it follows $r_i^{\mathrm{T}} W_i r_i = 0$. It is easy to see $r_i^{\mathrm{T}}(I_n - \frac{r_i r_i^{\mathrm{T}}}{r_i^{\mathrm{T}} r_i}) = 0$. Thus $\dot{V}_i = 0$, which implies $\|r_i(t)\| = \|r_i(0)\|$ for every $t \geqslant 0$. $\square$

By Proposition 1, we can see that, if $\|r_i(0)\| = 1$, then (1) becomes

$$\dot{r}_i = W_i r_i + k\left(I_n - r_i r_i^{\mathrm{T}}\right)\sum_{j=1}^{m} a_{ij} r_j. \tag{3}$$

If $\|r_i(0)\| \neq 0$, then the unit vector $\tilde{r}_i(t) = r_i(t)/\|r_i(t)\|$ satisfies

$$\dot{\tilde{r}}_i = W_i \tilde{r}_i + k\left(I_n - \tilde{r}_i \tilde{r}_i^{\mathrm{T}}\right)\sum_{j=1}^{m} b_{ij} \tilde{r}_j, \tag{4}$$

where $b_{ij} = a_{ij}\|r_j(0)\|/\|r_i(0)\|$.

Proposition 2. *If $n = 2$, $W_i = -W_i^{\mathrm{T}}$ and $\|r_i(0)\| = 1$, then the nonlinear multi-agent system (1) can be reduced to the classical Kuramoto model*

$$\dot{\theta}_i = \omega_i + k\sum_{j\in N_i}\sin(\theta_j - \theta_i). \tag{5}$$

Proof. The proof procedure is similar to that in [17]. Since $\|r_i(0)\| = 1$, we can consider (3). Let $r_i = (\cos\theta_i,\ \sin\theta_i)^{\mathrm{T}}$. Then

$$\dot{r}_i = \begin{pmatrix} -\sin\theta_i \\ \cos\theta_i \end{pmatrix}\dot{\theta}_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} r_i \dot{\theta}_i. \tag{6}$$

Since $W_i = -W_i^{\mathrm{T}}$, we can write $W_i = \begin{pmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{pmatrix}$. By (6) and (3), we have

$$\begin{aligned}\dot{\theta}_i &= r_i^{\mathrm{T}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\dot{r}_i\\ &= r_i^{\mathrm{T}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left(W_i r_i + k\left(I_2 - r_i r_i^{\mathrm{T}}\right)\sum_{j=1}^{m} a_{ij} r_j\right)\\ &= \omega_i + k\sum_{j=1}^{m} a_{ij} r_i^{\mathrm{T}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} r_j\\ &= \omega_i + k\sum_{j=1}^{m} a_{ij}\sin(\theta_j - \theta_i). \qquad \square \end{aligned} \tag{7}$$

Remark 1. Proposition 2 shows that (3) is a generalized form of Kuramoto model in high-dimensional linear spaces. Here we call (3) high-dimensional Kuramoto model. If we let $W_i = 0$, $k = 1$ and $\|r_i(0)\| = 1$ for $i = 1, 2, \dots, m$, then the high-dimensional Kuramoto model (3) is reduced to the system considered in [17].

Proposition 3. *Assume $W_i = W$ is skew symmetric for every $i = 1, 2, \dots, m$. Let $z_i(t) = e^{-Wt} r_i(t)$. Then $z_i(t)$ satisfies*

$$\dot{z}_i = k\left(I_n - z_i z_i^{\mathrm{T}}\right)\sum_{j=1}^{m} a_{ij} z_j. \tag{8}$$

Proof. Due to $W^{\mathrm{T}} = -W$, one can see $r_i^{\mathrm{T}} = z_i^{\mathrm{T}} \mathrm{e}^{W^{\mathrm{T}} t} = z_i^{\mathrm{T}} \mathrm{e}^{-Wt}$. Thus we have

$$\begin{aligned}\dot{z}_i &= -W \mathrm{e}^{-Wt} r_i + \mathrm{e}^{-Wt}\left(W r_i + k\left(I_n - r_i r_i^{\mathrm{T}}\right)\sum_{j=1}^{m} a_{ij} r_j\right)\\ &= k\left(I_n - z_i z_i^{\mathrm{T}}\right)\sum_{j=1}^{m} a_{ij} z_j. \qquad \square\end{aligned}$$

If $W_i = W$ for $i = 1, 2, \dots, m$, then, without loss of generality, we can assume the system is

$$\dot{r}_i = k\left(I_n - r_i r_i^{\mathrm{T}}\right)\sum_{j=1}^{m} a_{ij} r_j, \tag{9}$$

which can be rewritten as

$$\begin{aligned}&\dot{r}_i = k\left(I_n - r_i r_i^{\mathrm{T}}\right)\sum_{j=1}^{m} a_{ij}(r_j - r_i) \quad \text{or}\\ &\dot{r} = -kD(r)(L \otimes I_n) r, \end{aligned} \tag{10}$$

where $D(r) = \text{block-diag}\{I_n - r_1 r_1^{\mathrm{T}}, I_n - r_2 r_2^{\mathrm{T}}, \dots, I_n - r_m r_m^{\mathrm{T}}\}$ and $L = (l_{ij})$ being the Laplacian matrix of the interconnection graph.

Proposition 4. *Consider the high-dimensional Kuramoto model (9) limited on the unit sphere. The following statements hold:*

(1) *Vector $r = (r_1^{\mathrm{T}}, r_2^{\mathrm{T}}, \dots, r_m^{\mathrm{T}})^{\mathrm{T}} \in \mathbf{R}^{mn}$ is an equilibrium if and only if r_i is in parallel with $\sum_{j\in N_i} r_j$ denoted by $r_i /\!/ \sum_{j\in N_i} r_j$ for all $i = 1, 2, \dots, m$.*
(2) *Assume the interconnection graph is a strongly connected digraph. Then r is an equilibrium and there exists $p \in \mathbf{R}^n$ such that $p^{\mathrm{T}} r_i > 0$ for every $i = 1, 2, \dots, m$ if and only if $r_1 = r_2 = \cdots = r_m$.*