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Existence results for nano boundary layer flows with nonlinear Navier boundary condition

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ABSTRACT

The standard no slip boundary condition of classical fluid mechanics is no longer valid at the micro- and nano-scale and should be replaced by a boundary condition that allows some degree of tangential slip. In the present work, the classical laminar boundary layer equation of the flow away from the origin past a wedge with the no-slip boundary condition replaced by a nonlinear Navier boundary condition is revisited. This boundary condition includes an arbitrary index parameter, denoted by $n > 0$, which appears in the coefficients of the differential equation to be solved. It is proved corresponding to the value $n = \frac{1}{3}$, there are exactly three situations for the problem: (i) there is no solution; (ii) there exist two solutions; (iii) there exist four solutions. Furthermore, the exact analytical solution of the problem is given in terms of parabolic cylinder functions for further physical interpretations.

1. Preliminaries and problem formulation

The most theoretical investigations of the boundary layer equations have applied the no-slip boundary condition at the fluid–solid interface, which is a fundamental notion in fluid mechanics [1–4], and assumes that the fluid velocity component is zero relative to the solid boundary. This is not true for fluid flows at the micro- or nano-scale and the no-slip boundary condition does not apply but a certain degree of tangential velocity slip should be replaced [5,6]. Navier [7] proposed a boundary condition which states the component of the fluid velocity tangential to the boundary walls is proportional to the tangential stress. Afterwards, the linear Navier boundary condition, by some researchers [8–11], has been extended to a nonlinear form

$$|u| = l\left(\left|\frac{\partial u}{\partial y}\right|\right)^n, \tag{1}$$

where $l > 0$ is the constant slip length and $n > 0$ is an arbitrary power parameter. Consider the steady two-dimensional boundary layer equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = U\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{\partial^2 u}{\partial y^2}, \tag{3}$$

subject to the boundary conditions

$$|u| = l\left(\left|\frac{\partial u}{\partial y}\right|\right)^n \quad \text{and} \quad v = 0 \quad \text{at } y = 0, \tag{4}$$

$$u \equiv U(x) = ax^m \quad \text{as } y \to +\infty, \tag{5}$$

where x and y are the dimensionless Cartesian coordinates measured along the plate and normal to it. u and v are the velocity components along the x and y axes. U is a given external inviscid velocity field. The parameters a, m and n are constants. The case $a > 0$ is of

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main interest when describing flows away from the origin, and $a < 0$ when the external stream flows towards the origin (for more details see [12,13]). The more interesting problems arising out of finding a solution of boundary layer equation is when the investigation of the conditions under which two solutions are similar. Matthews and Hill [10,14] introduced the following similarity transformation, by using Lie symmetries analysis

$$\eta = x^{-\frac{n-1}{3n-2}} y, \tag{6}$$

and the stream function $f(\eta)$ defined by

$$\frac{u}{U} = \frac{1}{a} f', \qquad x^{\frac{2n-1}{3n-2}} \frac{v}{U} = \frac{1}{a}\left(\frac{n-1}{3n-2}\eta f' - \frac{2n-1}{3n-2}\eta f\right), \tag{7}$$

with $m = \frac{n}{3n-2}$ and $n \neq \frac{3}{2}$. By these transformations, Eqs. (2) and (3) can be written as

$$f''' + \frac{2n-1}{3n-2} f f'' - \frac{n}{3n-2}\left(f'^2 - a^2\right) = 0, \tag{8}$$

where the prime denotes differentiation with respect to η. The boundary conditions depend on the value of n. For $n \neq \frac{1}{2}$, the boundary conditions are

$$f = 0 \quad \text{and} \quad \left|f'\right| = l\left(\left|f''\right|\right)^n \quad \text{at } \eta = 0, \tag{9}$$

$$f' \to a \quad \text{as } \eta \to +\infty. \tag{10}$$

A further simplification shows the parameter a could be removed from the governing equation and the second boundary condition (10) if η and l are multiplied by $\sqrt{|a|}$ and $|a|^{\frac{3n-2}{2}}$, respectively; and f is multiplied by $\frac{1}{\sqrt{a}}$ for $a > 0$. Therefore, the differential equation of the flow away from the origin past a wedge is converted to

$$f''' + \frac{2n-1}{3n-2} f f'' - \frac{n}{3n-2}\left(f'^2 - 1\right) = 0, \tag{11}$$

with the boundary condition

$$f(0) = 0, \qquad \left|f'(0)\right| = l\left(\left|f''(0)\right|\right)^n, \qquad f'(+\infty) = 1. \tag{12}$$

In the next sections, we consider the above boundary value problem when $n = \frac{1}{3}$. In this case, Matthews and Hill [10] have shown that the solutions could be non-unique. We prove corresponding to the mentioned value of n for the problem (11) and (12), three situations are exactly occurred: (i) there is no solution for some values of the parameter l; (ii) the problem admits dual solutions for a specified critical value of l which will be determined later; (iii) there exist four solutions for some values of the parameter l. Furthermore, the exact analytical solution of the problem is given in terms of non-algebraic parabolic cylinder functions for further physical interpretations.

2. The exact analytical solution

Assuming $n = \frac{1}{3}$, the problem (11) and (12) is converted to the following one

$$f''' + \frac{1}{3} f f'' + \frac{1}{3}\left(f'^2 - 1\right) = 0, \tag{13}$$

with the boundary condition

$$f(0) = 0, \qquad \left|f'(0)\right| = l\sqrt[3]{\left|f''(0)\right|}, \qquad f'(+\infty) = 1. \tag{14}$$

One easily sees that (13) admits the first integral

$$f'' + \frac{1}{3} f f' - \frac{1}{3}\eta = C, \tag{15}$$

where C is the integral constant. Using the first boundary condition (14), Eq. (15) yields

$$C = f''(0) = \gamma. \tag{16}$$

Eq. (15) still admits the integral, then using notation (16), we obtain

$$f' + \frac{1}{6} f^2 - \frac{1}{6}\eta^2 = \gamma\eta + D, \tag{17}$$

where D is the integral constant. Again using the first boundary condition (14), Eq. (17) gives

$$D = f'(0) = \alpha. \tag{18}$$

Therefore, Eq. (17) is converted to

$$f' + \frac{1}{6} f^2 - \frac{1}{6}\eta^2 = \gamma\eta + \alpha. \tag{19}$$