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Entanglement of a two-particle Gaussian state interacting with a heat bath

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ABSTRACT

The effect of a thermal reservoir is investigated on a bipartite Gaussian state. We derive a pre-Lindblad master equation in the non-rotating wave approximation for the system. We then solve the master equation for a bipartite harmonic oscillator Hamiltonian with entangled initial state. We show that for strong damping the loss of entanglement is the same as for freely evolving particles. However, if the damping is small, the entanglement is shown to oscillate and eventually tend to a constant non-zero value.

1. Motivation

Entanglement is one of quantum mechanics' most fascinating features. It was first described in a celebrated paper by Einstein, Podolsky and Rosen [1] but owes its name to Schrödinger [2], who investigated its broader significance for the measurement question. It has taken on enhanced significance in quantum information. In this regard, the fragility of entanglement when the system is subjected to "outside" influence is of even greater importance. In the current work, we study a bipartite system with a Gaussian wave function. The state is prepared such that it is entangled, then shared between two parties who let their respective particle evolve either freely or interacting via a harmonic potential, but interacting with its own environment or heat bath. We study the resulting loss of entanglement between the particles. To do so, we use the pre-Lindblad non-rotating-wave master equation, [3,4], for which we outline a simple perturbative derivation starting with the Quantum Langevin Equation as derived in [5] and using a simple perturbation method as in [6].

The loss of entanglement in a system interacting with an environment is a well-known phenomenon. It has been studied in various systems, see e.g. [7–12], where it was found that there is often a sharp loss of entanglement when compared to a decoherence time scale, which has been termed entanglement sudden-death (E.S.D.). These studies are mainly in the context of qubits and the Rotating Wave Approximation (R.W.A.). The R.W.A. is obtained by discarding the fast oscillating terms in the equations of motion. This approximation works well for weak coupling and systems with well-spaced energy levels. However, we wish to consider a more general setting and as such this work presents a study of E.S.D. in a continuous-variables setting using the Non-Rotating-Wave (N.R.W.) approximation. Note that the master equation obtained in the N.R.W. approximation is not of the Lindblad form [13], hence does not in general satisfy the complete positivity condition. Yet, because the physical limits of the validity of this property are not well-understood [14], complete positivity alone does not ensure physicality of the result and one can easily check for the validity of the density matrix by checking its positive semi-definiteness. At the same time, the N.R.W. master equation often works better for systems which are strongly coupled to the environment [3]. Moreover, the unphysical behavior occurs for low temperatures only. Caldeira and Leggett [15] have derived a pre-Lindblad equation using a path-integrals method which is presumably not perturbative. We present a simple perturbative derivation of the N.R.W. master equation in Appendix A. Diósi [16,17] has generalist the Caldeira–Leggett derivation to obtain a more complicated equation which is valid for a range of low temperatures.

The choice of a continuous variables setting allows for a more realistic study of the evolution of the state of the chosen system. Gaussian states form a class of continuous variable states which is becoming more and more essential to the field of quantum optics. Indeed, their ease of experimental manipulation makes them very attractive for quantum information processing [18]. Gaussian states have also been widely studied analytically in the context of a system coupled to a heat bath, see e.g. [19–23] to cite but a few.

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In [23] in particular, Vasile et al. study two non-interacting quantum harmonic oscillators, coupled to two independent structured reservoir, examining various spectral densities for the bath. In [24], Ficek and Tanás study a system of two qubits coupled to a radiation field where they allow spontaneous decay of the atoms. They show that the entanglement vanishes but then is revived twice. In [25], the authors study the emergence of entanglement between two initially non-entangled qubits due to spontaneous emission, provided both atoms are initially excited and in the asymmetric state. Their results suggest that an interaction between two particles which are initially entangled can delay the vanishing of the entanglement and even revive it, or create entanglement between two initially non-entangled particles. We introduce a harmonic potential with frequency ω_0 as the interaction between the particles in our system and examine the dynamics of the entanglement. We show that entanglement revival can occur depending on the strength of the damping, i.e. how strong the coupling γ is with respect to the oscillator's frequency. We show that if the damping is small ($\gamma < 2\sqrt{2}\omega_0$), the entanglement eventually tends towards a limiting value and does not vanish asymptotically.

In Section 2 we recall the Langevin equation and present the main steps in the derivation of the master equation. We then recall, in Section 3, the formalism used to describe Gaussian states and the particular measure for entanglement we use. Section 4 considers free evolution, illustrating E.S.D. while Section 5 considers a harmonic interaction. Section 6 contains some concluding remarks.

2. Framework

In the following we outline very briefly a perturbative derivation of the N.R.W. Master Equation used here. Further details are in Appendix A. The derivation is given for one particle but generalizes easily to the case of two particles, each coupled to its own environment. We consider a heat bath modeled by independent oscillators coupled harmonically to the particle [5]. The corresponding Hamiltonian has the form

$$H = \frac{p^2}{2m} + V(x) + \frac{1}{2}\sum_j \left\{ \frac{p_j^2}{m_j} + m_j\omega_j^2 (q_j - x)^2 \right\}. \tag{1}$$

We will denote by H_s the Hamiltonian of the system alone, $H_s = \frac{p^2}{2m} + V(x)$. Solving the Heisenberg equations of motion for q_j and then for x yields the Quantum Langevin Equation (see [5])

$$m\ddot{x} + \int_{-\infty}^{t} \mu(t-t')\dot{x}(t')\,dt' + V'(x) = \xi(t), \tag{2}$$

where the dot denotes the derivative with respect to time and the prime on V that with respect to x. $\mu(t)$ and $\xi(t)$ describe the influence of the bath on the system and are known as the *memory function* and the operator-valued *random force* respectively and are expressed explicitly in Appendix A. In the case of an Ohmic heat bath, $\mu(t)$ effectively reduces to a constant γ. The Quantum Langevin Equation for a general observable Y of the small system (particle) then reads

$$\dot{Y} = \frac{\mathrm{i}}{\hbar}[H_s, Y] - \frac{\mathrm{i}}{2\hbar}\big[[x, Y], \xi(t)\big]_+ + \frac{\mathrm{i}\gamma}{2\hbar}\big[[x, Y], \dot{x}(t)\big]_+. \tag{3}$$

This equation is an equation for the system operators (Heisenberg representation), whereas a Master Equation is an (approximate) equation acting on the density operator of the quantum system under study (Schrödinger picture). The adjoint equation provides a link between the two formalisms:

$$\mathrm{Tr}\big\{Y(t)\rho\big\} = \mathrm{Tr}\big\{Y\rho(t)\big\}, \tag{4}$$

where Tr denotes the trace. Inserting (3), we obtain

$$\dot{\rho}(t) = -\frac{\mathrm{i}}{\hbar}\big[H_s, \rho(t)\big] - \frac{\mathrm{i}}{2\hbar}\big[\big[\xi(t), \rho(t)\big]_+, x\big] + \frac{\mathrm{i}\gamma}{2\hbar}\big[\big[\dot{x}, \rho(t)\big]_+, x\big]. \tag{5}$$

In order to derive the Master Equation from this adjoint equation, we assume that the bath is large and hence stays at thermal equilibrium, and that for $t \to -\infty$, the system and the bath are decoupled so that $\rho(t) \sim \rho_s(t)\rho_B$. This assumption is critical to the derivation of any Master Equation. Finally, assuming that the noise is small we write $\xi(t) \to \epsilon\xi(t)$, where ϵ is a small parameter. (This assumption is in fact not essential to the result but allows for a simpler derivation.) Applying a perturbation method and tracing over the bath yields the Non-Rotating-Wave Master Equation for $\rho_s(t)$ (see Appendix A)

$$\dot{\rho}_s(t) = -\frac{\mathrm{i}}{\hbar}\big[H_s, \rho(t)\big] + \frac{\mathrm{i}\gamma}{2\hbar}\big[\big[\dot{x}, \rho_s(t)\big]_+, x\big] - \frac{kT\gamma}{\hbar^2}\big[\big[\rho_s(t), x\big], x\big].$$

(In position space this equation agrees with (5.10) in [15].) This equation generalizes in an obvious way to the case of two particles, each in its own heat bath:

$$\begin{aligned}\dot{\rho}(t) = {} & -\frac{\mathrm{i}}{\hbar}\big[H_s, \rho(t)\big] \\ & + \frac{\mathrm{i}\gamma_1}{2\hbar}\big[\big[\dot{x}_1, \rho(t)\big]_+, x_1\big] + \frac{\mathrm{i}\gamma_2}{2\hbar}\big[\big[\dot{x}_2, \rho(t)\big]_+, x_2\big] \\ & - \frac{kT_1\gamma_1}{\hbar^2}\big[\big[\rho(t), x_1\big], x_1\big] - \frac{kT_2\gamma_2}{\hbar^2}\big[\big[\rho(t), x_2\big], x_2\big].\end{aligned} \tag{6}$$

(Here we have omitted the subscript s. γ_1 and γ_2 are the coupling parameters for the individual heat baths and T_1 and T_2 are the temperatures of the baths.)

3. Gaussian states and the logarithmic negativity

Since the states we will study are Gaussian, we now briefly recall the formalism for Gaussian states [26–28].

Gaussian states can be completely specified in terms of their first and second moments, described respectively by the displacement vector

$$d_j = \langle R_j\rangle_\rho = \mathrm{Tr}[R_j\rho]$$

and the covariance matrix

$$\Gamma_{j,k} = 2\,\mathrm{Re}\,\mathrm{Tr}\big[\rho\big(R_j - \langle R_j\rangle_\rho\big)\big(R_k - \langle R_k\rangle_\rho\big)\big]$$

where R is the vector $R^T = (q_1, p_1; \ldots; q_n, p_n)$; q_j and p_j are the canonical variables of a system of n oscillators with the usual canonical relations written as $[R_j, R_k] = \mathrm{i}\hbar\sigma_{jk}$ and σ a real skew-symmetric $2n \times 2n$ block matrix given by

$$\sigma = \bigoplus_{k=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The displacement vector is irrelevant in the study of entanglement and is taken to be zero in our examples. The covariance matrix thus reduces to

$$\Gamma_{j,k} = 2\,\mathrm{Re}\,\mathrm{Tr}[\rho R_j R_k]. \tag{7}$$

Any real symmetric positive-definite matrix A can be brought to its Williamson normal form [29] via symplectic transformations,