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Unified $(p, q; \alpha, \gamma, l)$-deformation of oscillator algebra and two-dimensional conformal field theory

I.M. Burban *

Bogolyubov Institute for Theoretical Physics, National Academy of Sciences of Ukraine, 14 b, Metrolohichna Str., 03680 Kyiv, Ukraine

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ABSTRACT

The unified $(p, q; \alpha, \gamma, l)$-deformation of a number of well-known deformed oscillator algebras is introduced. The deformation is constructed by imputing new free parameters into the structure functions and by generalizing the defining relations of these algebras. The generalized Jordan–Schwinger and Holstein–Primakoff realizations of the $U^{\alpha\gamma l}_{pq}(su(2))$ algebra by the generalized $(p, q; \alpha, \gamma, l)$-deformed operators are found. The generalized $(p, q; \alpha, \gamma, l)$-deformation of the two-dimensional conformal field theory is established. By introducing the $(p, q; \alpha, \gamma, l)$-operator product expansion (OPE) between the energy–momentum tensor and primary fields, we obtain the $(p, q; \alpha, \gamma, l)$-deformed centerless Virasoro algebra. The two-point correlation function of the primary generalized $(p, q; \alpha, \gamma, l)$-deformed fields is calculated.

1. Introduction

The quantum deformations of the universal enveloping algebra $U_q(\mathcal{L})$ of a simple Lie algebra $\mathcal{L}$ emerged in the study of solutions of the quantum Yang–Baxter equations [1]. V. Drinfeld extended this structure to general class associative algebras with the Hopf algebra structure, which is neither commutative nor co-commutative. The non-co-commutativity is achieved by the introducing of a formal parameter q. An example of such structures is the deformation of the universal enveloping algebra $U_q(su(2))$ [2,3]. The important tool in the study of these structures is their realization by the creation and annihilation operators (the Jordan–Schwinger and the Holstein–Primakoff constructions). In order to generalize the Jordan–Schwinger construction to quantum algebras, Biedenharn [4] and Macfarlane [5] introduced, independently, *the q-deformed creation and the annihilation operators*.

Long before another q-deformation of the canonical commutation relations has been used by Arik and Coon [6] for the operator description of the generalized Veneziano amplitude obtained by replacement of the Γ-function by the Γ_q-function.

Naturally, it would be desirable to generalize a particular mathematical structure as much as possible. In particular, it concerns the generalization of the one-parameter deformations of the Lie algebras, initiated in the works [7–10].

The (p, q)-deformed of the $U_{p,q}(su(2))$ algebra and its realization by the operators of the (p, q)-deformed oscillator algebra was studied in Refs. [11,12]. A construction two-parameter oscillator algebra also has been introduced in [13] under the name "Fibonacci" oscillator algebra.

The sequently investigations have shown that for algebraic structure of the (p, q)-deformed quantum $U_{p,q}(su(2))$-algebra the additional parameter is artificial and could be removed from it. On the level of oscillator algebra, the study of multi-parameter deformations of the oscillator algebra was continued in the works [14–18].

All these deformations are three-parameter generalizations of the one-parametric deformation of the Biedenharn–Macfarlane algebra. They yield the Jordan–Schwinger realizations of the generalized deformed Lie algebras $U_{p,q}(su(2))$.

The multi-parametric generalization of the two-parametric deformed oscillator algebra was considered [18,19]. Although algebraic structure of any multi-parameter deformed quantum algebra may be mapped one-to-one onto the standard one-parameter deformed algebra [20,21], its co-algebraic structure and physical results in both cases is not the same.

Many versions of the deformed oscillator algebras have been considered in the literature. Most of them can be embedded within the common mathematical framework of the *generalized deformed oscillator algebras*.

A generalized oscillator algebra is an associative algebra generated by the generators $\{1, a, a^+, N\}$, where a and a^+ are hermitian conjugate N self-adjoint operators, and the defining relations

* Fax: +380 44 526 59 98.
E-mail address: burban@bitp.kiev.ua.

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$$aa^+ = f(N+1), \qquad a^+a = f(N),$$
$$[N,a] = -a, \qquad \left[N,a^+\right] = a^+. \tag{1}$$

A positive analytic function $f(x) = [x]$ with $f(0) = 0$ is called *a structure function* [22]. This function defines a deformation scheme, and along with defining relations it entirely defines the deformed oscillator algebra. The most well-known structure function is the one of the two-parameter deformed oscillator algebra

$$f(x) = [x]_{pq} = \frac{p^{-x} - q^x}{p^{-1} - q}, \tag{2}$$

$p, q \in \mathbb{R}$ [11]. The structure function (2) and defining relations

$$aa^+ - q^{-1}a^+a = p^N, \qquad [N,a] = -a, \qquad \left[N,a^+\right] = a^+, \tag{3}$$

or

$$aa^+ - qa^+a = p^{-N}, \qquad [N,a] = -a, \qquad \left[N,a^+\right] = a^+ \tag{4}$$

define the (p,q)-deformed oscillator algebra [11–13]. This algebra describes the Arik–Coon $(p = 1, q)$ [6], the Biedenharn–Macfarlane $(p = q^\alpha, q^\gamma)$ [4,5], the Kwek, Oh $(p = q^\alpha, q^\gamma)$ [23] deformations of the oscillator algebras in unified framework.

The other version of the deformed oscillator algebra, different from (3) (or (4)), with the structure function

$$f(x) = [x]_q = \frac{q^{-x} - q^x}{q^{-1} - q} \tag{5}$$

and the defining relations

$$\left[a,a^+\right] = [N+1]_q - [N]_q, \qquad [N,a] = -a, \qquad \left[N,a^+\right] = a^+ \tag{6}$$

is the oscillator algebra, which in general is distinguished from the previous ones. The oscillator algebra (3) (or (4)) and its generalizations have not of the Hopf algebra structure whereas the (HY) algebra is endowed with this structure (HY) [24].

The number of the free parameters of the structure function of the deformed of oscillator algebra characterize a deformed scheme. One of the methods to obtain a multi-parameter deformed algebra is to add to the structure function several the new parameters. Naturally, the increasing of the number of the deformation parameters makes the application of the deformed oscillator algebras more flexible in the practical applications. In the framework of one-parameter deformation this scheme combined with scheme of construction of the generalized deformed oscillator algebras [22] has given possibility to unify in unified framework [14,15,23,16,18] the well-known deformations of the oscillator algebra [6,4,5].

These generalize deformed oscillator algebras have found the application in the concrete physical models [25,17]. The authors [25] have shown that the algebra of the observable system of two identical vortices in superfluid thin film is described by the generalized deformed algebra. The authors [17] considered this generalized deformed oscillator and have shown that Kerr medium transforms an ordinary linear harmonic oscillator into this generalized oscillator for specific values of the deformation parameters α, γ.

The different deformation schemes of the oscillator algebras have found their parallel reflection in the constructions of concrete deformed models of quantum physical systems. For example, the Jaynes–Cummings models in quantum optics, in parallel to deformed oscillator algebras, generated the Biedenharn–Macfarlane's [26], the f-oscillaor's [27], the (p,q)-deformed versions [11] of this model. The same takes place for two-dimensional conformal fields theories induced by the Arik–Coon [29], the Biedenharn–Macfarlane [30], the (p,q) [31], the Oh–Sigh [28] deformed versions of the oscillator algebras. In framework of the $(p,q;\alpha,\gamma,l)$-deformation these theories allow unification.

The q-deformed oscillator algebras were used in the construction of the three-dimensional non-commutative spaces with broken Lorentz invariance [32,33]. The $(p,q;\alpha,\gamma,l)$-deformed oscillator algebra may be useful in this area.

In this Letter we propose the unified deformation of the oscillator algebra which envelop as particular cases the well-known deformations [6,4,5,11,14,15]. The structure function ("generalized $(p,q;\alpha,\gamma,l)$-number") of this deformation

$$f(x) = [x]^{\alpha\gamma l}_{pq} = \frac{p^{-\alpha x} - q^{\gamma x}}{p^{-l/\gamma} - q^{l/\alpha}}, \tag{7}$$

where $\alpha, \gamma, l \in \mathbb{R}$, contains comparatively with (2) the additional deformation parameters.

Traditionally at the fixed structure function, the defining relation of the generalized deformed oscillator algebras are represented in the form (1). We generalize the defining relations (3) (or (4)) and (6) to obtain the various generalized deformed versions of the oscillators at the fixed structure function (7). We consider only generalized Daskaloyannis (GD), generalized Chakrabarti–Jagannathan (GCh-J), and generalized Hong Yang (GHY) versions of these algebras. The properties of these oscillator algebras in general are distinguished from each others. We study the relationship between them. The realization of the $U_q(su(2))$ algebra [2,3] by the q-deformed and (p,q)-deformed creation and annihilation operators independently has been done by different authors [4,5,11,12,23].

Generalizing the Jordan–Schwinger and the Holstein–Primakoff transformations we obtain the realizations $U^{\alpha\gamma l}_{pq}(su(2))$ algebra in terms of operators of the (GCh-J) and (GHY) deformed oscillator algebras. For convenience in the physical applications we included in formulas representations the non-zero Casimir invariants of the generalized deformed (GCh-J) and (GHY) oscillator algebras.

The above deformations of the oscillator algebra have found applications in the construction of the deformed two-dimensional integrable models, and the two-dimensional conformal field theory. Some of these deformations of the two-dimensional conformal field theories have been studied in the works [30,29,28,34]. The problem of the unification of these deformed theories arose also in this case.

We have considered the $(p,q;\alpha,\gamma,l)$-deformation of the two-dimensional conformal field theory. Note from the beginning, in sofa as the satisfactory central extension of the quantum deformed complex Witt algebra does not exists we restrict ourselves by the consideration of the centerless $(p,q;\alpha,\gamma,l)$-deformed Virasoro algebra and by the investigation of the two-dimensional conformal field theory based on it. In the following the $(p,q;\alpha,\gamma,l)$-deformed conformal field theory means the conformal field theory based on centerless $(p,q;\alpha,\gamma,l)$-deformed Virasoro algebra.

The conformal transformations of primary fields are generalized to $(p,q;\alpha,\gamma,l)$-deformed conformal transformations. We have found the pole structure of the $(p,q;\alpha,\gamma,l)$-deformed operator product expansion (OPE) of the holomorphic component of the energy–momentum tensor with primary fields. From this we have obtained centerless $(p,q;\alpha,\gamma,l)$-deformed Virasoro algebra. We have found the two-point correlation function of the $(p,q;\alpha,\gamma,l)$-deformed two-dimensional conformal field theory.

2. Some aspects of generalized $(p,q;\alpha,\gamma,l)$-deformed oscillator algebras

At the fixing structure function (7) we will consider the basic versions of the generalized deformed oscillator algebras and the relations between them.