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Maximum entropy principle for Kaniadakis statistics and networks

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A B S T R A C T

In this Letter we investigate a connection between Kaniadakis power-law statistics and networks. By following the maximum entropy principle, we maximize the Kaniadakis entropy and derive the optimal degree distribution of complex networks. We show that the degree distribution follows $P(k) = P_0 \exp_\kappa(-k/\eta_\kappa)$ with $\exp_\kappa(x) = (\sqrt{1+\kappa^2 x^2} + \kappa x)^{1/\kappa}$, and $|\kappa| < 1$. In order to check our approach we study a preferential attachment growth model introduced by Soares et al. [Europhys. Lett. 70 (2005) 70] and a growing random network (GRN) model investigated by Krapivsky et al. [Phys. Rev. Lett. 85 (2000) 4629]. Our results are compared with the ones calculated through the Tsallis statistics.

1. Introduction

Over the last years, a lot of effort has been dedicated to studies of *networks*. In this concern, a network can be defined as a mathematical abstraction created to represent a relationship between objects. Usually, the objects are called *nodes* and the relationships are called *edges*. The number of edges owned by a node is referred as the node's degree and the networks are classified according to a distribution of that degree (also called connectivity). Considering its definition, the network can be used to model a great quantity of natural and artificial systems [1].

Primordially, networks without an evident organization were described with the random graph theory introduced by Erdös and Rényi (ER) [2]. The associated model, ER model, gives rise to random networks whose connectivity distributions $P(k)$ are Poisson distributions.

The technological advances allowed to study larger amounts of data and new conclusions were found about the apparently disordered networks. In 1999, Albert et al. reported that the WWW links connectivity distribution obeys a power law [3] which could indicate a subjacent organization in that network. In order to try to understand the mechanisms that could lead to a non-evident order, Barabási and Albert (BA) [4] introduced a model that presents two ingredients: growth and preferential attachment. By considering this model, BA have shown that the $P(k)$ decays as a power law for large k, independently of the nature of the system [5].

On the other hand, as it is well known, some restrictions to the applicability of the standard statistical mechanics have motivated investigations of non-standard statistics, both from theoretical and experimental viewpoints. In fact, the Tsallis nonextensive statistical mechanics [6] and the generalized power-law statistics developed by Kaniadakis [7] are the most investigated frameworks. Several consequences (in different branches) of the former framework have been investigated in the literature [8], which include the study of Tsallis statistics in the context of complex networks [9–11]. In this concern, the Thurner–Tsallis model [9] shows that growth is not necessary for having scale-free degree distributions. The Kaniadakis statistics in turn is characterized by a κ-entropy that emerges naturally in the framework of the so-called kinetic interaction principle [7]. Several physical features of a κ-distribution have also been theoretically investigated [12].

In this Letter, by following the maximum q-entropy method in the context of complex networks [10,11], we derive an optimal degree distribution which maximizes the κ-entropy based on the Kaniadakis statistics [7]. As an application, we analyze the κ- and q-degree distributions in two scale-free network model, e.g. the preferential attachment growth [11] and the growing random network (GRN) model [13,14].

This Letter is organized as follows. A brief summary of Kaniadakis statistics is presented in Section 2. In Section 3, we present the maximum entropy method for the calculation of optimal degree distribution in the context of Kaniadakis framework.

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In Section 4, by using the preferential attachment growth model and growing random network (GRN) model, we investigate the κ- and q-degree distributions. We summarize our main conclusions in Section 5.

2. Kaniadakis framework

Recent studies on the kinetic foundations of the so-called κ-statistics led to the power-law distribution function and the κ-entropy which emerge naturally in the framework of the kinetic interaction principle (see, e.g., Ref. [7]). Formally, the κ-framework is based on the κ-exponential and the κ-logarithm functions defined as [7]

$$\exp_\kappa(x) = \left(\sqrt{1+\kappa^2x^2}+\kappa x\right)^{1/\kappa}, \tag{1}$$

$$\ln_\kappa(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa}, \tag{2}$$

with

$$\ln_\kappa\left(\exp_\kappa(x)\right) = \exp_\kappa\left(\ln_\kappa(x)\right) = x. \tag{3}$$

The κ-parameter belongs to the mathematical interval $|\kappa| < 1$ and in the case $\kappa = 0$ these expressions reduce to the usual exponential and logarithmic functions. The κ-entropy associated with the κ-framework is given by

$$S_\kappa = -\int d^3p f \ln_\kappa f \tag{4}$$

which fully recovers standard Boltzmann–Gibbs entropy, $S_{\kappa=0}(f) = -\int f \ln f d^3p$. As a matter of fact, the Kaniadakis entropy also can be a particular case of the Borges–Roditi entropy [15].

3. Maximum entropy method

3.1. Tsallis degree distributions

We recall the main aspects of the connection between the Tsallis statistics and complex networks. Specifically, the main result is the q-optimal degree distribution that maximizes the Tsallis entropy given by [6]

$$S_q = -\frac{1}{1-q}\left(1-\sum_i p_i^q\right), \tag{5}$$

where q represents the entropic index and p_i the probability distribution of the state i. Such entropy reduces to the Boltzmann–Gibbs–Shannon in the limit $q \to 1$. Here, the q-degree distribution reads [10,11]

$$P(k) = P_0 \exp_q\left(-\frac{k}{\eta_q}\right), \tag{6}$$

where $\eta_q > 0$ defines the characteristics number of links, k is the connectivity and the q-exponential function is defined as

$$\exp_q(x) \equiv \left[1+(1-q)x\right]^{\frac{1}{1-q}} \tag{7}$$

if $1+(1-q)x > 0$ and zero otherwise.

3.2. New approach

Now, let us discuss the standard method of maximization of the Kaniadakis entropy. Here and hereafter, the Boltzmann constant is set equal to unity for the sake of simplicity. Thus, the functional entropy to be maximized is

$$\delta S_\kappa^* = \delta\left(S_\kappa + \alpha\sum_k P(k) + \beta\sum_k kP(k)\right) \tag{8}$$

where α and β are the Lagrange multipliers. The Kaniadakis entropy is given by [7]

$$S_\kappa = -\frac{1}{2\kappa}\sum_k\left[\frac{1}{1+\kappa}P(k)^{1+\kappa} - \frac{1}{1-\kappa}P(k)^{1-\kappa}\right], \tag{9}$$

and the above constraints used are the normalization of the degree distribution and the averaged coordination number

$$\sum_k P(k) = 1 \quad \text{and} \quad \sum_k kP(k) = \langle k\rangle. \tag{10}$$

By considering the same arguments of Ref. [10], we derive, after some algebra, the following expression for the κ-degree distribution

$$P(k) = P_0 \exp_\kappa\left(-\frac{k}{\eta_\kappa}\right), \tag{11}$$

with

$$\exp_\kappa\left(-\frac{k}{\eta_\kappa}\right) = \left[\sqrt{1+\kappa^2\left(\frac{k}{\eta_\kappa}\right)^2} - \kappa\left(\frac{k}{\eta_\kappa}\right)\right]^{\frac{1}{\kappa}}. \tag{12}$$

Therefore, this new degree distribution, based on the Kaniadakis framework, is the power law that generalizes the exponential distribution. In particular, $\kappa \sim 0$ it behaves like the Tsallis degree distribution. Indeed, by using the asymptotic analytical behaviors of the q-exponential and κ-exponential functions, we obtain the following relation between the entropic parameters

$$\kappa = 1-q, \tag{13}$$

where the Gaussian limits $\kappa = 0$ and $q = 1$ are satisfied simultaneously in (13).

4. Applications

4.1. The preferential attachment growth model

4.1.1. Numerical model

In order to test the viability of the new degree distribution [Eq. (11)], let us consider the preferential attachment growth model. In this regard, we use the same model proposed in Ref. [11] that considered the following rules for the growing of lattice:

1. First, one site is fixed ($i = 1$) at some arbitrary origin of the plane.
2. The second site ($i = 2$) is randomly and isotropically chosen at a distance r distributed according to the probability law
$$P_G(r) \propto 1/r^{2+\alpha_G} \tag{14}$$
with $\alpha_G \geqslant 0$ (G stands for *growth*). This second site is then linked to the first one.
3. To locate the next sites ($i = 3, 4, 5, \dots, N$), the origin is moved to the barycenter of the existing sites and the distribution $P_G(r)$ is applied again from this new origin. The new site is now going to be linked to only one of the pre-existing sites in the lattice. To do this, it was used an attachment probability
$$p_A = \frac{k_i/r_i^{\alpha_A}}{\sum_{j=1}^{N-1} k_j/r_j^{\alpha_A}} \tag{15}$$
with $\alpha_A \geqslant 0$ (A stands for *attachment*), where r_i is the distance of the newly arrived site to the ith site of the pre-existing cluster, and the connectivity k_i is the number of links already arriving to the same ith site.