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Spooky action-at-a-distance in general probabilistic theories

Giacomo Mauro D'Ariano [a,b,*,1], Franco Manessi [a,1], Paolo Perinotti [a,b,1]

[a] QUIT group, Dipartimento di Fisica, via Bassi 6, 27100 Pavia, Italy
[b] INFN Gruppo IV, Sezione di Pavia, via Bassi, 6, 27100 Pavia, Italy

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ABSTRACT

We call a probabilistic theory "complete" if it cannot be further refined by no-signaling hidden-variable models, and name a theory "spooky" if every equivalent hidden-variable model violates Shimony's Outcome Independence. We prove that a complete theory is spooky if and only if it admits a pure steering state in the sense of Schrödinger. Finally we show that steering of complementary states leads to a Schrödinger's-cat-like paradox.

1. Introduction

Since the early days physicists have been wondering whether Quantum Theory (QT) can be considered complete [1,2], or more refined theories compatible with quantum predictions could exist. These models, also known as Hidden Variable Theories (HVT), reproduce QT thanks to a statistical definition of pure quantum states, which are obtained as averages over the more fundamental states of the HVT. In this approach, which reduces QT to a Statistical Mechanics, many results have been obtained, such as the theorems by Kochen–Specker and Bell [3,4], and the results by Conway–Kochen on the free will [5,6].

Recently, General Probabilistic Theories (GPT) have received great attention as the appropriate framework to study foundational aspects of physics [7–13]. Despite much work has been devoted to the relations between probabilistic theories and HVTs, these results are mostly a characterization of the probability measures, lacking a conceptual physical characterization of the theory itself, for example in terms of axioms. So far there exist examples of probability measures that do not respect locality, signaling, non-contextuality, determinism, completeness, etc., but none of these highlights the physical properties that a GPT must fulfill in order to achieve such violations.

The present Letter breaks the ground in the direction of providing a characterization theorem for complete "spooky" theories (see definitions in the following). Roughly speaking, the spookiness of a complete theory is the apparent "action at a distance" due to outcome correlations [14]. We show that spookiness for complete theories is equivalent to Schrödinger's steering property [15,16]. We do not discuss the completeness assumption since an exhaustive inquiry would require a much more complicate analysis, comparable to a generalized Bell theorem for GPTs. Finally, we use the results about spookiness to prove that complementarity and steering are necessary and sufficient conditions to raise a Schrödinger's-cat-like paradox.

2. Hidden variable theories for a GPT

The most important feature of a given probabilistic theory – such as QT or more generally any GPT – is the probability rule that links the various elements of the theory itself. More precisely, given a state ρ, a group of observers for the theory $(A, B, C, \dots)$ and the measurements $a, b, c, \dots$ that $A, B, C, \dots$ perform, the probability rule $\Pr[a_i, b_j, c_k, \dots | a, b, c, \dots \rho]$ is defined for every possible outcome $a_i, b_j, c_k, \dots$ over a suitable sample space Ω. In the remainder of the Letter, we will drop the explicit dependence of all probability rules on the state ρ. We can now define a hidden variable description for the previous model as follows.

Definition 1 *(Hidden variable theory).* An equivalent HVT for a GPT is given by a set $\Lambda \ni \lambda$, and a probability rule $\widetilde{\Pr}[\cdot|\cdot]$ on $\Omega \times \Lambda$, such that [17]

$$\Pr[a_i, b_j, c_k, \dots | a, b, c, \dots] = \sum_\lambda \widetilde{\Pr}[a_i, b_j, c_k, \dots | a, b, c, \dots, \lambda] \widetilde{\Pr}[\lambda | a, b, c, \dots] \quad (1)$$

for every state of the GPT.

* Corresponding author.
E-mail addresses: dariano@unipv.it (G.M. D'Ariano), franco.manessi01@ateneopv.it (F. Manessi), paolo.perinotti@unipv.it (P. Perinotti).
[1] http://www.quantummechanics.it.

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In the following we will restrict our attention to HVTs satisfying two requirements: λ-*independence*, namely $\widetilde{\Pr}[\lambda|a,b,c,\dots] = \widetilde{\Pr}[\lambda]$, i.e. λ is an *objective* parameter independent of the choice of measurements[2]; and *Parameter Independence*, namely $\widetilde{\Pr}[a_i|a,b,c,\dots,\lambda] = \widetilde{\Pr}[a_i|a,\lambda]$ and similarly for $b,c,\dots$, i.e. the HVT is *no-signaling*. Clearly, given a GPT, without these two restrictions we can always build an equivalent deterministic HVT which is signaling, and denies observers' free choice [17].

A GPT is *complete* if every equivalent HVT provides no further descriptive detail. Besides classical probability theory, there is at least a GPT that is complete in the present sense, which is indeed Quantum Theory, as proved recently by Colbeck and Renner in Ref. [19].

It is now crucial to require that probabilities depend non-trivially on the hidden variable.

Definition 2 *(Descriptively significant HVT).* A HVT is descriptively significant for an equivalent GPT if it satisfies λ-independence and Parameter Independence, and there exists a pure state and measurements $a,b,\dots,a_i,b_j,\dots$ such that for some $\lambda,\lambda' \in \Lambda$ with $\widetilde{\Pr}[a_i,b_j,\dots|a,b,\dots,\lambda] \neq 0$, one has

$$\widetilde{\Pr}[a_i,b_j,\dots|a,b,\dots,\lambda] \neq \widetilde{\Pr}[a_i,b_j,\dots|a,b,\dots,\lambda']. \quad (2)$$

Definition 3 *(Complete GPT).* A GPT is complete if every equivalent HVT is not descriptively significant.

The reason why it is important to investigate only descriptively significant HVTs is the following. Given a non-significant HVT for a given GPT, for all pure states and all $a,b,\dots,a_i,b_j,\dots$, we have that, by Eq. (2) and Eq. (1)

$$\Pr[a_i,b_j,\dots|a,b,\dots] = \widetilde{\Pr}[a_i,b_j,\dots|a,b,\dots,\lambda_i], \quad (3)$$

for all $\lambda_i \in \Lambda$ such that $\widetilde{\Pr}[a_i,b_j,\dots|a,b,\dots,\lambda_i] \neq 0$. Therefore, we conclude that $\widetilde{\Pr}[a_i,b_j,\dots|a,b,\dots,\lambda_i]$ shares all the features of $\Pr[a_i,b_j,\dots|a,b,\dots]$, e.g. non-locality or complementarity.

Given a GPT, among all HVTs equivalent to it and not descriptively significant, there is one theory that enjoys the so-called "*single-valuedness property*" [17].

Definition 4 *(Single-valuedness).* A HVT satisfies the single-valuedness property if $|\Lambda| = 1$.

For a HVT with single-valuedness there exists only one hidden variable value λ_0, whence for every i and j, $\Pr[a_i,b_j|a,b] \equiv \widetilde{\Pr}[a_i,b_j|a,b,\lambda_0]$. Given a GPT there is always an equivalent hidden variable model which satisfies single-valuedness [17]: this fact recalls the intuition that QT can be regarded itself as a HVT, where the hidden variable role is played by the quantum state. If we want to study a complete probabilistic theory it is useful to refer to the simplest non-descriptively significant equivalent hidden variable model, that is the one which satisfies single-valuedness.

Thanks to J.P. Jarrett [20], it is known that the Bell locality [3] is equivalent to the conjunction of two different properties: the aforementioned Parameter Independence and the Shimony's so-called Outcome Independence [21]. Parameter Independence corresponds to the property of "*no-signaling without exchange of physical systems*" in [9] for GPTs, while Outcome Independence can be stated as the factorizability of joint probabilities, i.e.[3]

$$\widetilde{\Pr}[a_i,b_j|a,b,\lambda] = \widetilde{\Pr}[a_i|a,b,\lambda] \times \widetilde{\Pr}[b_j|a,b,\lambda]. \quad (4)$$

Notice that the previous definition can be applied to a general GPT, regarded as a single-valued HVT.

The EPR paradox can be rewritten in the following similar way [22,17]: quantum predictions are not compatible with any equivalent non-descriptively significant HVT which satisfies Outcome Independence. For this reason, according to EPR, QT presents a *spooky action at a distance*. We now want to extend the EPR result, namely: which are the GPTs that present this spooky flavor? First we must define in what sense a theory can present spooky features.

Definition 5 *(Spooky theory).* A GPT is spooky if it violates Outcome Independence on a pure state and every equivalent descriptively significant HVT does so.

From now on, we will focus on complete spooky GPTs, unless told otherwise.

3. Review of general probabilistic theories

Before starting we need to introduce the usual notation for GPTs. For a detailed discussion see [7]. The symbols ρ_A, $|\rho)_A$ and $(\rho \,|\!\!\!\frac{A}{\ }$ denote the *state* ρ for system A, representing the information about the system initialization, including the probability that such preparation can occur. The set of the states of a given system A is a (truncated) positive cone, and therefore given the states $\{\rho_i\}_{i\in\eta}$ for A, every their convex combination belongs to the cone of the states of A. The extremal rays of the cone – namely the states which cannot be seen as a convex combination of other ones – are the so-called *pure* states.

Similarly, c_{iA}, $(c_i|_A$ and $\frac{A}{\ }\!\!\!|\, c_i)$ mean the *effect* c_i for system A or, in more practical terms, the i-th outcome of the test (measurement) $c = \{c_i\}_{i\in\eta}$ on system A. Given a system A, its effects are bounded linear positive functionals from the states of A to $[0,1] \subset \mathbb{R}$, and therefore they belong to the dual cone of the cone of the states. The application of the effect c_i on the state ρ is written as $(c_i|\rho)_A$ or $(\rho \,|\!\!\!\frac{A}{\ }\!\!\!|\, c_i)$ and it means the probability that the outcome of measure c performed on the state ρ of system A is c_i, i.e. $(c_i|\rho)_A := \Pr[c_i|c]$. In the following we will not specify the system when it is clear from the context or it is generic.

The symbol e_A will denote a *deterministic effect* for system A, namely a measurement with a single outcome. For any state σ, the symbol $(e|\sigma)$ denotes its preparation probability within a test including a measurement $\{c_i\}_{i\in\eta}$ such that $e = \sum_{i\in\eta} c_i$. A state σ is deterministic if we know with certainty that it has been prepared in any test, whence $(e|\sigma) = 1$ for every deterministic effect e. An *ensemble* is a collection of (possibly non-deterministic) states $\{\alpha_i\}_{i\in\eta}$ such that $\rho := \sum_{i\in\eta} \alpha_i$ is deterministic. A GPT is causal (i.e. it satisfies the *no-signaling from the future* axiom [7]) iff the deterministic effect is unique. Thanks to this last feature, in a causal GPT

[2] Notice that a realistic theory where λ is correlated with the observers' choices could in principle be considered, however such a theory would be necessarily *ad hoc*, and even more puzzling than its original GPT [18].

[3] The usual definition of *Outcome Independence* in the literature is the following. A probabilistic HVT satisfies the Outcome Independence property if and only if $\forall a,b,c,\dots,a_i,b_j,c_k,\dots,\lambda$ on

$$\Pr[a_i|a,b,c,\dots,b_j,c_k,\dots,\lambda] = \Pr[a_i|a,b,c,\dots,\lambda],$$

$$\Pr[b_j|a,b,c,\dots,b_j,c_k,\dots,\lambda] = \Pr[b_j|a,b,c,\dots,\lambda],$$

$$\Pr[c_k|a,b,c,\dots,b_j,c_k,\dots,\lambda] = \Pr[c_k|a,b,c,\dots,\lambda],$$

and so on. One can easily prove that this definition is equivalent to Eq. (4).