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Synchronization of self-sustained oscillators inertially coupled through common damped system

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ABSTRACT

We study the dynamics of two self-oscillating systems inertially coupled to a linear oscillator. This interaction mechanism results in various types of synchronous motions such as in-phase, anti-phase and phase synchronization. We demonstrate the existence of mono-stable regimes and multi-stable behavior with two or more coexisting attractors. We present the bifurcational analysis revealing transition mechanisms between these regimes. In the multi-stable case, we examine the role of coupling parameter and shape of oscillations (the parameter indicating nonlinearity and strength of the damping) in various structure formations of attraction basins.

1. Introduction

Synchronization of coupled self-oscillating systems is of significant interest both from applied and fundamental points of view. This special type of collective rhythmicity can occur in various natural and artificial systems and can lead to dramatic consequences [1–3]. Currently, there are a lot of examples for constructive applications of this effect: in engineering it is used for improvement of the line-width of a high power generator with the help of a low power generator, having a narrower spectral line. It is gained a great importance as a mean to generate high power laser sources [4]. On the other hand, synchronously vibrating objects can cause various destructions of the whole system, pathologically synchronous firing generated in different brain areas is the hallmark of epileptical seizures [5]. Therefore, for the proper use of positive properties of synchronization phenomenon and reduction of its negative impact, it is important to understand the mechanisms leading to this type of behavior.

In recent years, there has been a growing interest in studying of synchronization in systems where the direct connection between the interacting elements is absent. In such systems the elements affect each other through an additional media (through a resonator [6], a parallel RLC-load [7], a bath representing the concentration of melatonin in the bloodstream [8], a feedback loop [9], etc.). In particular, there has been a lot of works on the detailed analysis of synchronization phenomena in Huygens-like systems, where the crucial interaction comes from movements of the common frame supporting the pendulum clocks [10]. Both from the theoretical and the experimental points of view, these systems have been studied by Blekhman [1], Bennett et al. [11], Pantaleone [12], Pogromsky et al. [13], Fradkov et al. [14], Senator [15], Oud et al. [16], Graichen et al. [17], Czolczynski et al. [18]. In these works, various mathematical models have been proposed to reproduce Huygens original results. Note, that in his original experiment, Huygens found that the pendulum clocks swung in anti-phase only.

In order to explain the existence of synchronized motion, various approximate approaches have been used. Particularly, the synchronization problem in [11] was theoretically studied by deriving a Poincare map for nonlinear dynamics. Simulations of this map revealed three possible types of attracting states: anti-phase oscillations, quasi-periodic state, where the pendula run at different frequencies, and so-called "beating death" state, in which one or both clocks cease to run.

The existence of the in-phase oscillations in Huygens-like setup was firstly observed and explained in [1] by means of coupled Van der Pol equations. Later, neglecting the damping of the base motion and deriving approximate evolution equations by the method of averaging, the stability diagrams for the in-phase and anti-phase synchronization states have been obtained in [12]. However, the peculiarities of these approximate approaches does not allow obtaining the complete picture of possible regimes. Note that the important feature of the in-phase synchronization is N^2-increase of the common load oscillations power with N-increase of the

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number of oscillators, that has received special attention in other applications [6,19].

In the present work we systematically analyze a system of two self-sustained oscillators coupled to a linear load. It can be demonstrated that the equations of this type can be obtained from the Huygens problem by means of shortened expansions [20]. In order to show the dynamical features of the considered system, various types of coexisting attractors and their basins of attraction had been numerically analyzed in [20,21]. It has also been shown that in-phase, anti-phase and phase synchronization are possible here. The main aim of the present work is to reveal the conditions and bifurcational mechanisms for the emergence of these regimes, to study the effect of the coupling strength and shape of self-oscillations on the transition from mono-stable (with one type of synchronous motion) to multi-stable regimes.

Note that coupling scheme considered in this work, was also previously analyzed by Peles and Wiesenfeld [7] for various electronic arrays. By expanding solutions of the considered system in terms of a small parameter, the authors obtained an analytically tractable iterative map for the derivation of the stability boundary for the in-phase state. This introduction of the small parameter, however, implies that only quasi-harmonic oscillatory motions can be considered within the framework of this approach. In the present work different shapes of oscillations (from quasi-periodic to relaxation forms) are considered. The bifurcational mechanisms of transition from one type of synchronization to another with the change of this form are revealed.

Moreover, the richest dynamics have been found for Huygens-like system in recent experimental study of Oud et al. [16]. It has been shown that in these systems more types of synchronization can be observed. Besides the anti-phase and in-phase motions of metronomes, some intermediate regimes of oscillations are possible. However, there is no clear understanding of mechanisms, leading to appearance of these regimes. In this work, the emergence of attractors lying outside of both in-phase and anti-phase manifolds is demonstrated within the framework of the considered system. It is shown that oscillations corresponding to these attractors are phase-synchronized, i.e. the phase difference for their oscillations is always bounded.

The Letter is organized as follows. In Section 2 we show the outline of the theoretical approach used in the present work. Further, the dynamics of two indirectly coupled Van der Pol–Duffing oscillators inside the two-dimensional anti-phase manifold and the stability conditions for the anti-phase oscillations are discussed (Section 3.1). The existence and stability of the in-phase regime are examined in Section 3.2. Section 4 presents the results obtained from numerical integration of the considered equations. Finally, in Section 5 the obtained results are summarized and discussed.

2. Outline of theoretical approach

Let us consider a system that is described by the following equations:

$$\ddot{x}_i + \Phi(x_i, \dot{x}_i) = -\delta\ddot{y}, \quad i = 1, 2,$$
$$\ddot{y} + h\dot{y} + \Omega^2 y = \sum_{i=1}^{2} \Phi(x_i, \dot{x}_i), \tag{1}$$

where parameters h and Ω are a damping factor and eigenfrequency of the linear oscillator, respectively; δ is a coupling parameter. We assume that $\Phi(-x, -\dot{x}) = -\Phi(x, \dot{x})$. In this case the system (1) possess two symmetries defining the invariance under the maps

$$\Theta: \quad (x_1, x_2, y) \to (x_2, x_1, y)$$
$$(\dot{x}_1, \dot{x}_2, \dot{y}) \to (\dot{x}_2, \dot{x}_1, \dot{y}) \tag{2}$$

and

$$\Psi: \quad (x_1, x_2, y) \to (-x_1, -x_2, -y)$$
$$(\dot{x}_1, \dot{x}_2, \dot{y}) \to (-\dot{x}_1, -\dot{x}_2, -\dot{y}). \tag{3}$$

The map (2) defines the mirror symmetry with respect to the four-dimensional in-phase manifold $M_s := \{(x_1, \dot{x}_1) = (x_2, \dot{x}_2)\}$, while the transformation (3) gives the central symmetry. It should be noted that, the transformations Θ and Ψ are commutative of each other, i.e. $\Theta \circ \Psi = \Psi \circ \Theta$, and their combination

$$\Theta \circ \Psi: \quad (x_1, x_2, y) \to (-x_2, -x_1, -y)$$
$$(\dot{x}_1, \dot{x}_2, \dot{y}) \to (-\dot{x}_2, -\dot{x}_1, -\dot{y}) \tag{4}$$

also gives the symmetry for the system (1).

Note that, the existence of the two-dimensional anti-phase manifold $M_a := \{(x_1, \dot{x}_1) = (-x_2, -\dot{x}_2), \ \ y = \dot{y} = 0\}$ follows from Eqs. (1) [22].

We assume that $\Phi(x, \dot{x}) = \phi(x) + \lambda\psi(x)\dot{x}$. In this particular case we have the Lienard equation for modeling the oscillating behavior of each single element. It is known that the conditions for the existence of oscillations in this case are: $\phi(-x) = -\phi(x)$, $\phi'(x) > 0$, $\psi(x) < 0$ for $|x| < x_0$ and $\psi(x) > 0$ for $|x| > x_0$. In general case, the function $\psi(x)$ may also depend on the derivative, i.e. $\psi(x, \dot{x})$.

We are interested in the stability of synchronous regimes observed in the system (1). Particularly, to study the stability of the in-phase and anti-phase oscillations, it is convenient to define the sum and difference variables:

$$\xi = \frac{x_1 + x_2}{2}, \qquad \eta = \frac{x_1 - x_2}{2}. \tag{5}$$

As a result, Eqs. (1) can be rewritten in the form

$$\ddot{\xi} + \frac{1}{2}\big[\Phi(\xi - \eta, \dot{\xi} - \dot{\eta}) + \Phi(\xi + \eta, \dot{\xi} + \dot{\eta})\big] = -\delta\ddot{y},$$
$$\ddot{\eta} + \frac{1}{2}\big[\Phi(\xi + \eta, \dot{\xi} + \dot{\eta}) - \Phi(\xi - \eta, \dot{\xi} - \dot{\eta})\big] = 0,$$
$$\ddot{y} + h\dot{y} + \Omega^2 y = \Phi(\xi - \eta, \dot{\xi} - \dot{\eta}) + \Phi(\xi + \eta, \dot{\xi} + \dot{\eta}). \tag{6}$$

The stability of the in-phase solution is determined by the linearized variational equations in a vicinity of the in-phase manifold $M_s := \{\eta = 0, \ \dot{\eta} = 0\}$

$$\ddot{\eta} + \Phi_\eta(\xi, \dot{\xi})\eta + \Phi_{\dot{\eta}}(\xi, \dot{\xi})\dot{\eta} = 0, \tag{7}$$

which is driven by the dynamics of the equations on M_s:

$$\ddot{\xi} + \Phi(\xi, \dot{\xi}) = -\delta\ddot{y},$$
$$\ddot{y} + h\dot{y} + \Omega^2 y = 2\Phi(\xi, \dot{\xi}). \tag{8}$$

Similarly, the stability conditions for the anti-phase solution are derived from the following set of equations:

$$\ddot{\eta} + \Phi(\eta, \dot{\eta}) = 0,$$
$$\ddot{\xi} + (1 + 2\delta)\big[\Phi_\xi(\eta, \dot{\eta})\xi + \Phi_{\dot{\xi}}(\eta, \dot{\eta})\dot{\xi}\big] = \delta\big(h\dot{y} + \Omega^2 y\big),$$
$$\ddot{y} + h\dot{y} + \Omega^2 y = 2\big[\Phi_\xi(\eta, \dot{\eta})\xi + \Phi_{\dot{\xi}}(\eta, \dot{\eta})\dot{\xi}\big], \tag{9}$$

where the oscillating solution on anti-phase manifold $M_a := \{\xi = 0, \ \dot{\xi} = 0, \ y = 0, \ \dot{y} = 0\}$ given from the first equation of (9), plays a role of the master oscillator.

3. Van der Pol–Duffing oscillators coupled to a common load

In order to model the oscillating behavior of self-oscillating systems, we consider $\Phi(x_i, \dot{x}_i) = \omega^2 x_i + \lambda(x_i^2 - 1)\dot{x}_i - \alpha x_i^3$, $i = 1, 2$. In Huygens-like systems, for example, the second term models the escapement mechanism of the pendulum clocks, and the cubic