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A chaotic system with a single unstable node

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ABSTRACT

This paper describes an unusual example of a three-dimensional dissipative chaotic flow with quadratic nonlinearities in which the only equilibrium is an unstable node. The region of parameter space with bounded solutions is relatively small as is the basin of attraction, which accounts for the difficulty of its discovery. Furthermore, for some values of the parameters, the system has an attracting torus, which is uncommon in three-dimensional systems, and this torus can coexist with a strange attractor or with a limit cycle. The limit cycle and strange attractor exhibit symmetry breaking and attractor merging. All the attractors appear to be hidden in that they cannot be found by starting with initial conditions in the vicinity of the equilibrium, and thus they represent a new type of hidden attractor with important and potentially problematic engineering consequences.

1. Introduction

Most familiar examples of low-dimensional chaotic flows occur in systems having one or more saddle points. Such saddle points allow homoclinic and heteroclinic orbits and the prospect of rigorously proving the chaos when the Shilnikov condition is satisfied. Furthermore, such saddle points provide a means for locating any strange attractors by choosing an initial condition on the unstable manifold in the vicinity of the saddle point. Such attractors have been called "self-excited," and they are overwhelmingly the most common type described in the literature.

Recently, many new chaotic flows have been discovered that are not associated with a saddle point, including ones without any equilibrium points, with only stable equilibria, or with a line containing infinitely many equilibrium points [1–10]. The attractors for such systems have been called "hidden attractors" [11–17], and that accounts for the difficulty of discovering them since there is no systematic way to choose initial conditions except by extensive numerical search. Hidden attractors are important in engineering applications because they allow unexpected and potentially disastrous responses to perturbations in a structure like a bridge or aircraft wing.

Here we introduce a new class of hidden attractor that occurs in a system in which the only equilibrium is an unstable node, and we identify what may be the simplest example of such a system with a strange attractor. By "unstable node" we mean an equilibrium point whose eigenvalues are all real and positive. The system was found by extensive numerical search and appears to be extremely rare in the class of system studied, but it has a number of interesting and unusual properties including symmetry breaking, attractor merging, and multistability, as well as an attracting torus.

Section 2 describes the numerical search procedure, and Section 3 describes the properties of the equilibrium point. Section 4 shows the variety of different dynamics and their bifurcations. Section 5 illustrates examples of coexisting attractors, and Section 6 provides the evidence that these attractors are hidden. Finally, Section 7 gives the conclusions.

2. Numerical search

Perhaps the simplest chaotic flow with a single equilibrium point is a jerk system, the most general quadratic form of which is given by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= f(x,y,z) = a_1 x + a_2 y + a_3 z + a_4 y^2 \\ &\quad + a_5 z^2 + a_6 xy + a_7 xz + a_8 yz + a_9 \end{aligned} \tag{1}$$

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System (1) has only one equilibrium point at $(-\frac{a_9}{a_1}, 0, 0)$ with eigenvalues λ that satisfy

$$\lambda^3 - f_z\lambda^2 - f_y\lambda - f_x = 0 \tag{2}$$

where $f_x = a_1$, $f_y = a_2 - a_6a_9/a_1$, and $f_z = a_3 - a_7a_9/a_1$. Since the Routh–Hurwitz stability criterion gives the conditions for having the real part of all eigenvalues negative, a transformation of Eq. (2) with $\lambda \to -\lambda$ gives the conditions for having them all positive. Thus Eq. (2) becomes $-\lambda^3 - f_z\lambda^2 + f_y\lambda - f_x = 0$, which after multiplying by -1 gives

$$\lambda^3 + f_z\lambda^2 - f_y\lambda + f_x = 0 \tag{3}$$

Thus the conditions for the equilibrium to be unstable are $f_z > 0$, $f_x + f_y f_z < 0$, and $f_x > 0$, or

$$\begin{aligned}
&\left(a_3 - \frac{a_7a_9}{a_1}\right) > 0\\
&a_1 + \left(a_2 - \frac{a_6a_9}{a_1}\right)\left(a_3 - \frac{a_7a_9}{a_1}\right) < 0\\
&a_1 > 0
\end{aligned} \tag{4}$$

An extensive numerical search involving millions of random combinations of the coefficients a_1 through a_9 and initial conditions subject to the constraints in Eq. (4) did not reveal any bounded solutions with a largest Lyapunov exponent greater than 0.001. Thus it seems likely that system (1) does not admit chaotic solutions in the presence of such a fully unstable equilibrium.

Therefore, inspired by the Sprott A (Nose–Hoover) system [18–20], and using a form that has successfully given other chaotic flows with hidden attractors [1,2,5], system (1) was modified slightly according to

$$\begin{aligned}
\dot{x} &= y\\
\dot{y} &= -x + yz\\
\dot{z} &= a_1x + a_2y + a_3z + a_4x^2 + a_5y^2\\
&\quad + a_6xy + a_7xz + a_8yz + a_9
\end{aligned} \tag{5}$$

System (5) has only one equilibrium point at $(0, 0, -a_9/a_3)$ with eigenvalues λ that satisfy

$$\lambda^3 - \left(a_3 - \frac{a_9}{a_3}\right)\lambda^2 + (-a_9 + 1)\lambda - a_3 = 0 \tag{6}$$

Changing the variable $\lambda \to -\lambda$ gives

$$\lambda^3 + \left(a_3 - \frac{a_9}{a_3}\right)\lambda^2 + (1 - a_9)\lambda + a_3 = 0 \tag{7}$$

The Routh–Hurwitz stability criterion guarantees that the real part of all eigenvalues are positive provided

$$\begin{aligned}
&\left(a_3 - \frac{a_9}{a_3}\right) > 0\\
&\left(a_3 - \frac{a_9}{a_3}\right)(1 - a_9) - a_3 > 0\\
&a_3 > 0
\end{aligned} \tag{8}$$

For this case, many chaotic solutions were found in an extensive computer search, although they are still relatively rare. Perhaps the simplest such system [21] is given by

$$\begin{aligned}
\dot{x} &= y\\
\dot{y} &= -x + yz\\
\dot{z} &= z + ax^2 - y^2 - b
\end{aligned} \tag{9}$$

with an appropriate choice of the parameters a and b and initial conditions. System (9) satisfies the conditions of Eq. (8) provided $b > 0$. With seven terms, this is actually a three-parameter system, but for simplicity, the third parameter is taken as unity. The remainder of the paper is concerned with the properties of system (9).

3. Equilibrium properties

By design, this system (9) has only one equilibrium at $(0, 0, b)$ with eigenvalues λ given by

$$\lambda^3 - (b+1)\lambda^2 + (b+1)\lambda - 1 = 0 \tag{10}$$

whose roots are $\lambda = 1$, $b/2 \pm \sqrt{b^2 - 4}$. Since one of the eigenvalues is $+1$, the equilibrium is always unstable, but the type of equilibrium depends on b and is independent of a as summarized in Table 1 where $\Delta = b^2 - 4$ and $\omega = \sqrt{4 - b^2}$.

Chaotic solutions occur for $b > 1$ and are most abundant at large b where the equilibrium is an unstable node. For a typical value of $b = 4$, the eigenvalues are $\lambda_1 = 3.732050808$, $\lambda_2 = 1$, $\lambda_3 = 0.267949192$, and the corresponding eigenvectors are

$$\begin{aligned}
\mathbf{v}_1 &= \pm(k_x + 3.732050808k_y)\\
\mathbf{v}_2 &= \pm k_z\\
\mathbf{v}_3 &= \pm(k_x + 0.267949192k_y)
\end{aligned} \tag{11}$$

Table 1
Types of equilibrium points for different values of the parameter b.

Parameter b	Eigenvalues	Type of equilibrium point
$b < -2$	1, $(b \pm \sqrt{\Delta})/2$	saddle node
$-2 < b < 0$	1, $(b \pm i\omega)/2$	saddle focus
$0 < b < 2$	1, $(b \pm i\omega)/2$	unstable focus
$b > 2$	1, $(b \pm \sqrt{\Delta})/2$	unstable node

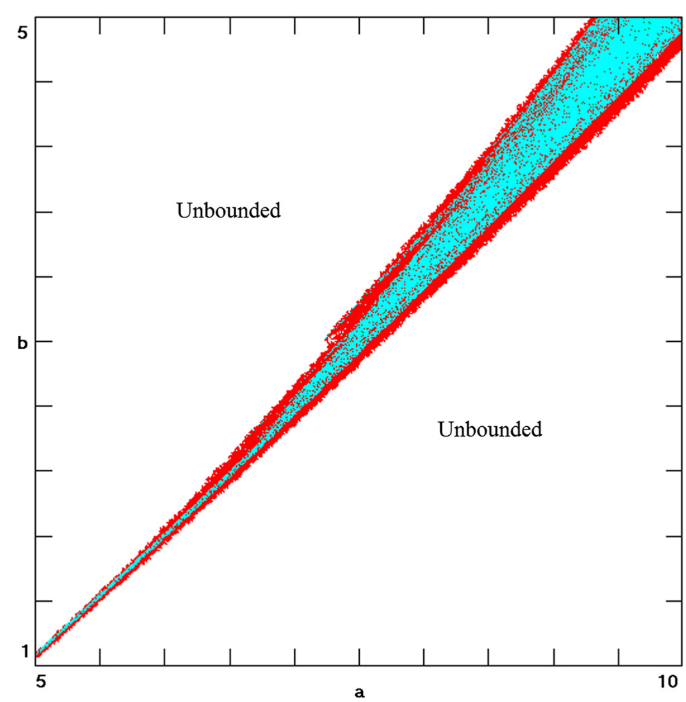

Fig. 1. Regions of various dynamical behaviors for system (9) as a function of the bifurcation parameters a and b. The chaotic regions are shown in red, the periodic (limit cycle) and quasiperiodic (torus) regions are shown in blue, and the unbounded regions are shown in white. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)