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Fractal electrodynamics via non-integer dimensional space approach

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ABSTRACT

Using the recently suggested vector calculus for non-integer dimensional space, we consider electrodynamics problems in isotropic case. This calculus allows us to describe fractal media in the framework of continuum models with non-integer dimensional space. We consider electric and magnetic fields of fractal media with charges and currents in the framework of continuum models with non-integer dimensional spaces. An application of the fractal Gauss's law, the fractal Ampere's circuital law, the fractal Poisson equation for electric potential, and equation for fractal stream of charges are suggested. Lorentz invariance and speed of light in fractal electrodynamics are discussed. An expression for effective refractive index of non-integer dimensional space is suggested.

1. Introduction

Fractal electrodynamics based on continuum models of fractal distribution of charges, currents and fields has been suggested in [1–5] ten years ago. These continuum models use the concept of power-law density of states and an application of fractional-order integration. It has been proved that D-order integration is connected with D-dimensional integration [4]. Then these continuum models of fractal electrodynamics have been applied and developed in two directions: (a) fractional integral models by Baskin and Iomin [6,7], by Ostoja-Starzewski [8] to describe anisotropic fractal cases; (b) fractional (non-integer) dimensional models by Muslih, Baleanu and coauthors [9–11], by Zubair, Mughal, Naqvi [12–16], by Balankin with coauthors [17], to describe an anisotropic case, multipoles, and electromagnetic waves in fractional space. Effective continuum models of fractal electrodynamics, which is considered in papers [9–17], are based on Stillinger and Palmer–Stavrinou generalizations of the scalar Laplacian that are suggested in [18] and [19], respectively. In these papers [18,19], the authors have proposed only the second order differential operators for scalar fields in the form of the scalar Laplacian in the non-integer dimensional space. The first order operators such as gradient, divergence, curl operators, and the vector Laplacian are not considered in [18,19].

Possibility to use only the scalar Laplacian in non-integer dimensional space approach greatly restricts us in application of continuum models of fractal media. For example, Stillinger's form of Laplacian cannot be used for the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic fields $\mathbf{B}(\mathbf{r}, t)$ in electrodynamic continuum models with non-integer dimensional spaces.

In recent paper [21], it was suggested a generalization of vector differential operators of first orders (gradient, divergence, curl operators) and the vector Laplacian for non-integer dimension spaces. This allows us to extend the scope of possible applications of continuum models with non-integer dimensional spaces. Using this new tool we can describe isotropic fractal media by using the non-integer dimensional space approach.

For anisotropic fractal case, an attempt to suggest D-dimensional vector operations of first order has been presented in the works [12–17]. In these papers, the gradient, divergence, and curl operators are suggested only as approximations of the square of the Palmer–Stavrinou form of Laplace operator. Recently [22] a generalization of gradient, divergence, and curl has been suggested without any approximation. The strict approach to continuum models of anisotropic fractal media by the vector calculus on non-integer dimensional space has been described in [22], where a review of different approaches are also suggested.

It should not be confused fractal electrodynamics and fractional electrodynamics that is based on fractional-order vector calculus [23]. Note that first time the fractional calculus has been applied in the electrodynamics by Joseph Liouville about two hundred years ago [24]. An attempt to use a fractional calculus in electrodynamics by introducing some differential vector operations was made by Engheta [25–28]. In these papers, the fractional integral vector operations and fractional generalization of integral theorems of Green, Stokes and Gauss are not considered. A rigorous self-consistent formulation of fractional differential and integral vector calculus was suggested in [23]. The fractional-order differential and integral vector operations are mutually agreed by the

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use of the Caputo fractional derivatives as an inverse operation to the Riemann–Liouville fractional integration. Using this inter-consistency of fractional differential and integral vector operators, the fractional Green's, Stokes' and Gauss's theorems have been are proved. We can note that the theory of fractional-order derivatives and integrals has be applied to several specific electromagnetic problems (for example, see [29–37]).

It should be also noted that the term "fractal electrodynamics" is used in the narrow sense in engineering [38–40] to describe fractal antennas, arrays and apertures and electromagnetic wave scattering from fractal surfaces. We use this term in a broader sense to the theory of fractal distribution of charges, currents, fields, and to electrodynamics of fractal media, and electromagnetic fields on fractal sets.

In this paper, we demonstrate an application of the vector calculus on non-integer dimensional space, which is suggested in [21], to fractal electrodynamics in isotropic case. We give an application of the fractal Gauss's law, the fractal Ampere's circuital law, the fractal Poisson equation for electric potential, and equation for fractal stream of charges.

2. *D*-dimensional integration and differentiation

Let us give some introduction to noninteger-dimensional integration and differentiation of integer orders (for details, see [18–22]).

The D-dimensional integration (see [18] and Section 4 of [20]) for scalar functions $f(\mathbf{r}) = f(|\mathbf{r}|)$ can be defined in terms of ordinary integration by the expression

$$\int d^D\mathbf{r}\, f(\mathbf{r}) = \int_{\Omega_{D-1}} d\Omega_{D-1} \int_0^\infty dr\, r^{D-1}\, f(r), \tag{1}$$

where we can use

$$\int_{\Omega_{D-1}} d\Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)} = S_{D-1}. \tag{2}$$

For integer $D = n$, equation (2) gives the well-known area S_{n-1} of $(n-1)$-sphere with unit radius.

As a result, the explicit expressions [20] of D-dimensional integration for arbitrary non-integer D has the form

$$\int d^D\mathbf{r}\, f(|\mathbf{r}|) = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty dr\, r^{D-1}\, f(r). \tag{3}$$

This equation reduced D-dimensional integration to ordinary one-dimensional integration. It is obvious that the linearity and translation invariance follow from linearity and translation invariance of ordinary integration. The scaling and rotation covariance can also be derived from equation (3).

In the continuum models of fractal media, it is convenient to work with the physically dimensionless variables $x/R_0 \to x$, $y/R_0 \to x$, $z/R_0 \to x$, $\mathbf{r}/R_0 \to \mathbf{r}$, that yields dimensionless integration and dimensionless differentiation in D-dimensional space. In this case the physical quantities of fractal media have correct physical dimensions.

The volume of D-dimensional ball V_D of radius R Is given by the expression

$$|V_D| = \frac{\pi^{D/2}}{\Gamma(D/2+1)} R^D, \tag{4}$$

and surface area of the d-dimensional sphere S_d of radius R is given by

$$|S_d| = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)} R^d. \tag{5}$$

In general, the dimension d of the boundary $S_d = \partial V_D$ of the region V_D of fractal medium and the dimension D of the region V_D are not related by the equation $d = D-1$. The difference between D and d defines a radial dimension $\alpha_r = D - d$ of the fractal medium. If the radial dimension is equal to one, then (5) can be represented by

$$|S_d| = \frac{2\pi^{D/2}}{\Gamma(D/2)} R^{D-1}. \tag{6}$$

The vector differential operators for non-integer dimension have been derived in [21] by analytic continuation in dimension from integer n to non-integer D.

For simplification we will consider two following cases:

1) *Spherically symmetric case of fractal media*, where scalar field φ and vector fields $\mathbf{E}$, $\mathbf{B}$ are independent of angles

$$\varphi(\mathbf{r}) = \varphi(r), \quad \mathbf{E}(\mathbf{r}) = E_r(r)\,\mathbf{e}_r, \quad \mathbf{B}(\mathbf{r}) = B_r(r)\,\mathbf{e}_r,$$

where $\mathbf{e}_r = \mathbf{r}/r$, $r = |\mathbf{r}|$ and $E_r = E_r(r)$ $B_r = B_r(r)$ are the radial component of $\mathbf{E}$ and $\mathbf{B}$. In this case, we will work with rotationally covariant functions only. This simplification is analogous to the simplification of integration over non-integer dimensional space suggested in [20].

2) *Axially (cylindrical) symmetric case of fractal media*, where the fields $\varphi(r)$ and $\mathbf{E}(r) = E_r(r)\,\mathbf{e}_r$, $\mathbf{B}(r) = B_r(r)\,\mathbf{e}_r$ are also axially symmetric. We assume that z-axis is directed along the axis of symmetry [21].

In [21], the equations of differential operators for non-integer D have been proposed in the following forms, where $m = 1$ and $m = 2$ describe spherically and axially (cylindrical) symmetric cases, respectively.

The divergence in non-integer dimensional space for the vector field $\mathbf{E} = \mathbf{E}(r)$ is

$$\mathrm{Div}^D_r\, \mathbf{E} = \frac{\partial E_r(r)}{\partial r} + \frac{D-m}{r} E_r(r). \tag{7}$$

The gradient in non-integer dimensional space for the scalar field $\varphi = \varphi(r)$ is

$$\mathrm{Grad}^D_r\, \varphi = \frac{\partial \varphi(r)}{\partial r}\, \mathbf{e}_r. \tag{8}$$

The curl operator for the vector field $\mathbf{E} = \mathbf{E}(r)$ is equal to zero, $\mathrm{Curl}^D_r\, \mathbf{E} = 0$.

The scalar Laplacian in non-integer dimensional space for the scalar field $\varphi = \varphi(r)$ is

$${}^S\Delta^D_r \varphi = \mathrm{Div}^D_r\, \mathrm{Grad}^D_r\, \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{D-m}{r}\frac{\partial \varphi}{\partial r}. \tag{9}$$

The vector Laplacian in non-integer dimensional space for the vector field $\mathbf{E} = E_r(r)\,\mathbf{e}_r$ is

$$\begin{aligned} {}^V\Delta^D_r \mathbf{E} &= \mathrm{Grad}^D_r\, \mathrm{Div}^D_r\, \mathbf{E} \\ &= \Big(\frac{\partial^2 E_r(r)}{\partial r^2} + \frac{D-m}{r}\frac{\partial E_r(r)}{\partial r} - \frac{D-m}{r^2} E_r(r) \Big)\, \mathbf{e}_r. \end{aligned} \tag{10}$$

For $D = 3$ equations (7)–(10) give the well-known expressions for the gradient, divergence, scalar Laplacian and vector Laplacian in $\mathbb{R}^3$ for fields $\varphi = \varphi(r)$ and $\mathbf{E}(r) = E_r(r)\,\mathbf{e}_r$.

The suggested operators allow us to reduce D-dimensional vector differentiations (7)–(10) to usual derivatives with respect to $r = |\mathbf{r}|$. As a result, we can reduce partial differential equations