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Theory of frequency synchronization in a ring laser

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ABSTRACT

The self-consistent problem of the frequency synchronization of counter-propagating waves in a ring laser is rigorously solved. An intrinsic nonlinear mechanism of the phase coupling between the waves is considered for the first time. This ineradicable coupling is provided by modulation of the population difference of the energy levels of the active medium atoms in the electromagnetic field of two counter-propagating waves. The theoretical limit for the range of phase locking between the counter-propagating waves is established. The general equation of phase synchronization is obtained from the solution of a self-consistent problem. The frequency-dependent boundaries of the synchronization band calculated in the framework of this approach show good agreement with experimental results published in the literature.

1. Introduction

Synchronization or phase locking is a well-known phenomenon common to all coupled oscillators. The interest in investigating this fundamental phenomenon in quantum systems is increased due to the rapid development of ring lasers. Nowadays ring lasers are widely used in high precision experimental research in various fields of applied and fundamental physics.

In the cavity of a ring laser two light waves traveling in the opposite directions are generated. In the idealized situation of identical conditions for the counter-propagating waves both the frequencies and intensities of these waves are the same. The equality of the frequencies is violated if some source of anisotropy is introduced, say, by imposing a magnetic field or by rotating the cavity. This asymmetry causes a beat frequency $\Delta\omega$, which can be measured with a high precision. However, in case of small perturbations, that is, of small $\Delta\omega$ values, the accuracy of measurements is restricted by the effect of synchronization. The great majority of publications devoted to investigation of this phenomenon are related to one of the most important applications of the ring laser, namely to its usage as a laser gyroscope sensor (see, for example, [1–14]).

The counter-propagating waves in a ring laser can be regarded as a system of coupled auto-oscillators. We can investigate the general phenomenon of synchronization in ring lasers on the example of a laser gyroscope without loss of generality. Our choice of this special system is due to the fact that almost all known experimental results regarding the locking of oscillators' frequencies refer to measurements of ring lasers' characteristics.

The operation of such devices is based on the Sagnac effect. When a ring cavity is rotated about the axis perpendicular to the cavity's plane, a certain difference $\Delta\omega$ between the frequencies of the counter-propagating waves appears which is proportional to the angular velocity Ω of rotation: $\Delta\omega = \mu\Omega$. The value of μ factor depends on the device parameters: $\mu = 4S/(\lambda L)$, where L is the cavity perimeter (the optical path), S is the area enveloped by the loop. This expression can be written in an equivalent form by using the momentary phase difference φ of the counter-propagating waves:

$$\Delta\omega = \dot{\varphi}. \tag{1}$$

Here the dimensionless scale factor μ is included in $\dot{\varphi}$. If the laser gyroscope rotates with an angular velocity smaller than some critical value called the lock-in threshold, the mutual synchronization of the frequencies is observed [1] – the beat frequency equals zero.

The phenomenon of the counter-propagating waves phase-locking in a ring laser is caused by their mutual coupling. In the literature (see, for example, [2–13]) the principal (and sometimes the sole) source of coupling is assigned to mutual scattering of each beam's energy in the direction of the other. While constructing a mathematical model of a ring laser, the existence of a linear in amplitude coupling between the counter-propagating waves is usually assumed. This coupling can arise if some portion of one wave is scattered in the direction of the other by an occasional tiny obstacle. The scattered field is considered in [2–13] as a source in the wave equations for the fields of the counter-propagating waves:

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$$-\frac{\partial^2 E_j}{\partial z^2}+\frac{\omega}{c^2 Q_j}\frac{\partial E_j}{\partial t}+\frac{1}{c^2}\frac{\partial^2 E_j}{\partial t^2}+\mu\frac{\partial^2 E_j}{\partial z\partial t}$$
$$=\frac{\omega_j^2}{c^2}P_j-\bar{r}_{j'}\frac{\partial E_{j'}}{\partial t}, \quad (2)$$

where $j, j' = r, l$ (r, l denote the waves propagating clockwise and counterclockwise, respectively), Q_j is the quality factor of the cavity for j-wave, P_j is the polarization of the nonlinear active medium induced by the field of j-wave, $\bar{r}_j = r_j \exp\{i\varepsilon_j\}$ are complex reflection indexes.

Such equations for the counter-propagating waves are usually solved by successive approximations, and an expression for the beat frequency is obtained already in the zeroth approximation (see, for example, [3–5]):

$$\dot{\psi}=\Omega+(c/L)\{r_r(E_r/E_l)\sin(\psi-\varepsilon_r)$$
$$+r_l(E_l/E_r)\sin(\psi+\varepsilon_l)\}. \quad (3)$$

The relevance of this approach, which leads to the phase-locking of the counter-propagating waves even in a passive cavity (in the absence of generation), will be discussed elsewhere. Here we only note that the above-mentioned conclusion regarding the phase-locking in a passive cavity contradicts to the commonly accepted definition of the synchronization effect [15]. Synchronization is usually understood as adjusting of frequencies and phases of two (or more) auto-oscillators by virtue of a weak interaction among them. It is a fundamental *nonlinear* phenomenon, which arises as a result of interaction of *auto-oscillating* systems. A ring cavity becomes an auto-oscillating device (a laser) only if the pumping energy exceeds the losses of one or several modes of the cavity. In this case the properties of oscillations (their shape, amplitude, and frequency) are determined by the system itself, and do not depend on the initial conditions.

The aim of this publication is a discussion of the entirely nonlinear mechanism of the phase coupling between the generated counter-propagating waves, which appears only when the oscillating system is involved into a nonlinear regime of self-sustained oscillations. To explain the phase coupling between the counter-propagating waves, we have no need in commonly used arbitrary artificial assumption of scattering some portion of energy from one wave to the other by imperfections of the cavity, which in principle can be eliminated. The natural mechanism of coupling discussed in this paper establishes theoretical limit to the width of the lock-in interval. It is provided by the parametric modulation in space and time of populations of the active medium energy levels, which is caused by the light wave. This modulation brings to life the waves of polarization on combined frequencies and strongly influences regimes of the ring laser radiation.

2. The problem setting

We consider a ring laser with an intercavity cell of length H containing a non-uniformly broadened active medium. According to the experiments in which the phenomenon of lock-in was investigated, we restrict ourselves to a single-mode approximation. This means that the waves propagating clockwise and counterclockwise have the same longitudinal index, but their frequencies ω_j and amplitudes $E_j(\mathbf{r}, t)$ may differ from one another ($j = r, l$). We represent the intercavity electromagnetic field as a superposition of the counter-propagating waves' fields:

$$\mathcal{E}(\mathbf{r},t)=\mathcal{E}_r(\mathbf{r},t)+\mathcal{E}_l(\mathbf{r},t)$$
$$=E_r(\mathbf{r})\exp\{-i\omega_r t+i\phi_{0r}\}+E_l(\mathbf{r})\exp\{-i\omega_l t+i\phi_{0l}\}$$
$$+\text{c. c.} \quad (4)$$

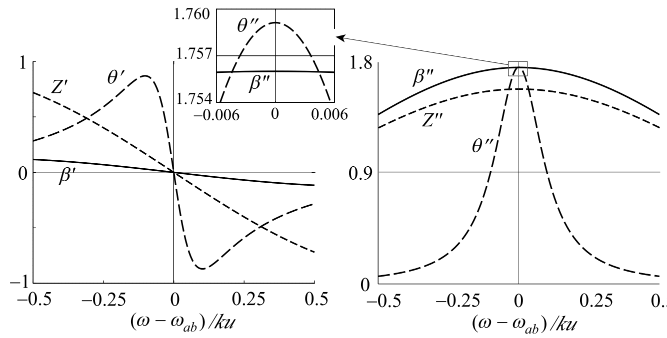

Fig. 1. Dependencies of real (left) and imaginary (right) parts of polarization coefficients Z, β and θ on detuning for the case of equal frequencies of the counter-propagating waves (single isotope).

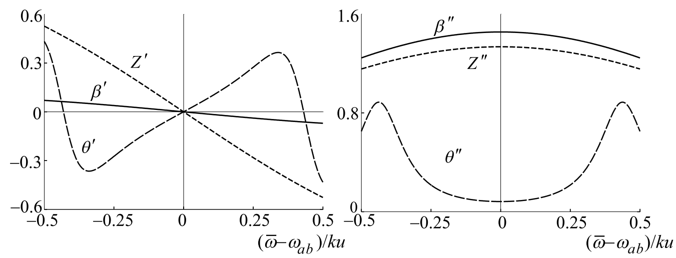

Fig. 2. Dependencies of real (left) and imaginary (right) parts of polarization coefficients Z, β, and θ on detuning (two isotopes).

where $E_j(\mathbf{r}, t)$ are slowly varying amplitudes of the fields, and ϕ_{0j} are their initial phases ($j = r, l$).

The field inside the cavity satisfies the wave equation

$$\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathcal{E}(\mathbf{r},t)=4\pi\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathcal{P}(\mathbf{r},t), \quad (5)$$

where Δ is the Laplace operator.

In a ring laser with rather large perimeter of the cavity, a single-mode generation is possible only at a sufficiently small excess of pumping over the threshold value and, as a consequence, at small radiation intensity. This allows us to restrict the calculation of polarization to the third order of the perturbation theory. In this case

$$\mathcal{P}(\mathbf{r},t)=P_r(\mathbf{r})\exp\{-i\omega_r t\}+P_l(\mathbf{r})\exp\{-i\omega_l t\}$$
$$+P_{lr}(\mathbf{r})\exp\{-i(2\omega_l-\omega_r)t\}$$
$$+P_{rl}(\mathbf{r})\exp\{-i(2\omega_r-\omega_l)t\}+\text{c. c.} \quad (6)$$

We represent the polarization at frequencies ω_r, ω_l as follows:

$$2\pi P_j(\mathbf{r})\exp\{-i\omega_j t\}$$
$$=(1/k)K\left\{Z_j-\beta_j I_j-\theta_{j'}I_{j'}\right\}E_j(\mathbf{r})\exp\{-i\omega_j t\}+\text{c. c.} \quad (7)$$

The specific form of polarization coefficients is analytically calculated with the help of the traditional scheme of solving the system of equations for the elements of the density matrix [17]. All the calculations are performed for the first time in a closed form (without approximations). Here we present the graphs of real and imaginary parts of the coefficients for the case of equal frequencies of the counter-propagating waves (see Figs. 1, 2). The calculations are performed for the following values of parameters: $\lambda = 0.6328$ μm; the half-widths of the homogeneous line of the transition $\gamma_{ab} = 104$ MHz; the half-widths of the transition energy levels a and b are $\gamma_a = 14$ MHz, $\gamma_b = 35$ MHz respectively; Doppler broadening $ku = 1000$ MHz. For the case of 50%-mixture of ^{20}Ne