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Linear superposition for a class of nonlinear equations

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ABSTRACT

We demonstrate a kind of linear superposition for a large number of nonlinear equations which admit elliptic function solutions, both continuum and discrete. In particular, we show that whenever a nonlinear equation admits solutions in terms of Jacobi elliptic functions $\mathrm{cn}(x,m)$ and $\mathrm{dn}(x,m)$, then it also admits solutions in terms of their sum as well as difference, i.e. $\mathrm{dn}(x,m) \pm \sqrt{m}\,\mathrm{cn}(x,m)$. Further, we also show that whenever a nonlinear equation admits a solution in terms of $\mathrm{dn}^2(x,m)$, it also has solutions in terms of $\mathrm{dn}^2(x,m) \pm \sqrt{m}\,\mathrm{cn}(x,m)\,\mathrm{dn}(x,m)$ even though $\mathrm{cn}(x,m)\,\mathrm{dn}(x,m)$ is not a solution of that nonlinear equation. Finally, we obtain similar superposed solutions in coupled theories.

1. Introduction

Linear superposition principle is one of the hallmarks of linear theories which does not hold good in nonlinear theories because of the nonlinear term. For example, even if two solutions are known for a nonlinear theory, their superposition is in general not a solution of the nonlinear theory. The purpose of this Letter is to point out a kind of superposition for a large number of nonlinear equations. In particular, there are several nonlinear equations, both discrete and continuum [1–3], which are known to admit exact periodic solutions in terms of Jacobi elliptic functions $\mathrm{cn}(x,m)$ as well as $\mathrm{dn}(x,m)$, where m denotes the modulus of the elliptic function [4]. Many of these solutions have found application in several areas of physics [5,6]. In particular, we have examined a large number of nonlinear equations, both continuum and discrete, which admit both $\mathrm{cn}(x,m)$ and $\mathrm{dn}(x,m)$ solutions and find that in all these cases even $\mathrm{dn}(x,m) \pm \sqrt{m}\,\mathrm{cn}(x,m)$ is also an exact solution. We have also examined a number of *coupled* field theories and have obtained superposed coupled solutions of the form $\mathrm{dn} \pm \sqrt{m}\,\mathrm{cn}$ in both the fields.

Further, we have also examined a number of continuum field theories which admit $\mathrm{dn}^2(x,m)$ as a solution and find that such theories also admit $\mathrm{dn}^2(x,m) \pm \sqrt{m}\,\mathrm{cn}(x,m)\,\mathrm{dn}(x,m)$ as solutions. While this cannot be treated as a linear superposition of two solutions (since $\mathrm{cn}(x,m)\,\mathrm{dn}(x,m)$ is not a solution of these models), we find it rather remarkable that such solutions exist, without exception, in a number of continuum field theories (including several coupled models) that we have looked at so far.

In this Letter we only discuss a few selected examples, several other examples will be given elsewhere in a longer version [7]. To begin with, we discuss one continuum (quadratic–cubic nonlinear Schrödinger equation or QCNLS) and one discrete model (saturated discrete nonlinear Schrödinger equation or DNLS) both of which admit $\mathrm{cn}(x,m)$ as well as $\mathrm{dn}(x,m)$ as solutions and show that these models also admit $\mathrm{dn}(x,m) \pm \sqrt{m}\,\mathrm{cn}(x,m)$ as solutions. Note that stable kink [8] and chaotic soliton solutions under nonlinearity management [9] are known for the QCNLS equation which arises in such diverse physical systems as nonlinear optics, chemical kinetics, matter-radiation interactions and mathematical ecology [8,9]. The DNLS equation also arises in many physical contexts [3,10]. We then discuss a coupled NLS–MKdV model and show that it has $\mathrm{dn}(x,m) \pm \sqrt{m}\,\mathrm{cn}(x,m)$ type coupled solutions in both the fields. Here MKdV refers to the modified KdV equation [5]. Note that the new solutions are not connected by any Landen transformation.

Subsequently, we discuss the Korteweg–de Vries (KdV) equation which is known to admit $\mathrm{dn}^2(x,m)$ as a solution [5,6] and show that this model also admits $\mathrm{dn}^2(x,m) \pm \sqrt{m}\,\mathrm{cn}(x,m)\,\mathrm{dn}(x,m)$ as solutions even though $\mathrm{cn}(x,m)\,\mathrm{dn}(x,m)$ is not a solution of the KdV equation. We also show that the coupled NLS–MKdV model, not only admits $\mathrm{dn}(x,m) \pm \sqrt{m}\,\mathrm{cn}(x,m)$ but also $\mathrm{dn}^2(x,m) \pm \sqrt{m}\,\mathrm{cn}(x,m)\,\mathrm{dn}(x,m)$ type solutions in both the fields. Further, we show that a coupled NLS–KdV model has superposed solutions of the form $\mathrm{dn}(x,m) \pm \sqrt{m}\,\mathrm{cn}(x,m) - \mathrm{dn}^2(x,m) \pm \sqrt{m}\,\mathrm{cn}(x,m)\,\mathrm{dn}(x,m)$. Finally, we briefly mention possible reasons why such a linear superposition works in several nonlinear theories. Note that only the KdV equation discussed below is an integrable system. We do have similar results for other integrable models such as NLS and MKdV, which will be presented in a longer version [7]. To the best of our knowledge, all solutions reported below are completely new.

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2. $\mathrm{dn} \pm \sqrt{m}\,\mathrm{cn}$ as exact solutions

We first discuss the quadratic–cubic nonlinear Schrödinger equation (QCNLSE) and then the discrete NLS equation (DNLSE) and show the existence of such superposed solutions in both cases.

2.1. QCNLSE

We start with the quadratic–cubic nonlinear Schrödinger equation

$$iu_t + u_{xx} + g_1|u|u + g_2|u|^2u = 0. \tag{1}$$

One of the exact moving periodic solutions to this equation is

$$u = \big(A\,\mathrm{dn}\big[\beta(x - vt + \delta_1), m\big] + B\big)\exp\big[-i(\omega t - kx + \delta)\big], \tag{2}$$

provided

$$g_2A^2 = 2\beta^2, \qquad g_2B^2 = (2-m)\beta^2, \qquad g_1 = -3Bg_2,$$
$$\omega = k^2 + 2(2-m)\beta^2, \qquad v = 2k. \tag{3}$$

Here δ, δ_1 are two arbitrary constants.

Similarly, another exact moving periodic solution to the QCNLSE, Eq. (1), is

$$u = \big(A\sqrt{m}\,\mathrm{cn}\big[\beta(x - vt + \delta_1), m\big] + B\big) \times \exp\big[-i(\omega t - kx + \delta)\big], \tag{4}$$

provided

$$g_2A^2 = 2\beta^2, \qquad g_2B^2 = (2m-1)\beta^2, \qquad g_1 = -3Bg_2,$$
$$\omega = k^2 + 2(2m-1)\beta^2, \qquad v = 2k. \tag{5}$$

Notice that the $\mathrm{cn}(x,m)$ solution only exists if $1/2 < m \leqslant 1$. We now show that, remarkably, even though $\mathrm{cn}(x,m)$ solution does not exist if $m \leqslant 1/2$, a linear superposition of $\mathrm{cn}(x,m)$ and $\mathrm{dn}(x,m)$ is still an exact solution over the entire range $0 < m \leqslant 1$, i.e.

$$u = \left(\frac{A}{2}\,\mathrm{dn}\big[\beta(x - vt + \delta_1), m\big] + \frac{D}{2}\sqrt{m}\,\mathrm{cn}\big[\beta(x - vt + \delta_1), m\big] + B\right) \times \exp\big[-i(\omega t - kx + \delta_2)\big], \tag{6}$$

is an exact solution to the QCNLSE, Eq. (1), provided

$$D = \pm A, \qquad g_2A^2 = 2\beta^2, \qquad g_2B^2 = (1+m)/2\beta^2,$$
$$g_1 = -3Bg_2, \qquad \omega = k^2 + (1+m)\beta^2, \qquad v = 2k. \tag{7}$$

Here $\mathrm{cn}(x,m)$ and $\mathrm{sn}(x,m)$ are periodic functions with period $4K(m)$, $\mathrm{dn}(x,m)$ is a periodic function with period $2K(m)$, with $K(m)$ being the complete elliptic integral of the first kind [4]. It is worth noting that the frequency ω of the three solutions (i.e. cn, dn and $\mathrm{dn} \pm \sqrt{m}\,\mathrm{cn}$) is different except at $m = 1$. We thus have two new periodic solutions of QCNLSE depending on if $D = A$ or $D = -A$.

Solutions (2) and (4): For given values of g_1, and g_2, the solutions depend on 7 parameters A, B, k, β, v, ω and m. Since there are 5 relations between them one has a two-parameter family of solutions. This is over and above the dependence of the solution on two arbitrary parameters δ and δ_1 due to translational invariance. The latter is true for all of our solutions and hence we will not mention such dependence in other solutions. The solution (6) depends on 8 parameters. Since there are 6 relations between these, once again we have a two-parameter family of solutions. Note that while solution (2) may or may not cross zero depending on if $A > 0$ or $A < 0$, the solutions (4) and (6) definitely cross zero twice in one period.

Few remarks are in order here which are in fact valid for all the models (both continuum and discrete) that admit such solutions.

1. Both the solutions $\mathrm{dn} + \sqrt{m}\,\mathrm{cn}$ and $\mathrm{dn} - \sqrt{m}\,\mathrm{cn}$ exist for the same values of the parameters.
2. In the limit $m = 1$, the three solutions dn, cn as well as $\mathrm{dn} + \sqrt{m}\,\mathrm{cn}$ go over to the well-known pulse (i.e. sech) solution.
3. In all the continuum models admitting such solutions, the factors of $2 - m$ or $2m - 1$ which appear in the dn and cn solutions, get replaced by the factor of $(1+m)/2$ in the $\mathrm{dn} \pm \sqrt{m}\,\mathrm{cn}$ solutions.
4. On the other hand, in the discrete theories admitting such solutions (see below), the factor of $\mathrm{dn}(\beta,m)$ or $\mathrm{cn}(\beta,m)$ appearing in dn and cn solutions, gets replaced by a factor of $[\mathrm{dn}(\beta,m) + \mathrm{cn}(\beta,m)]/2$ in the $\mathrm{dn} \pm \sqrt{m}\,\mathrm{cn}$ solutions.
5. In view of the above two points, either the frequency ω or the velocity v (or could be even both) are different for dn, cn and $\mathrm{dn} \pm \sqrt{m}\,\mathrm{cn}$ solutions except at $m = 1$.

2.2. Saturated discrete NLS equation

We now consider a discrete nonlinear equation which admits both dn and cn solutions and show that it also has superposed solutions $\mathrm{dn} \pm \sqrt{m}\,\mathrm{cn}$. In particular, we consider the saturated DNLS equation which has received great attention in many physical contexts [3,10]

$$i\frac{du_n}{dt} + [u_{n+1} + u_{n-1}] + \frac{\nu|u_n|^2}{1 + |u_n|^2}u_n = 0. \tag{8}$$

One well-known periodic solution to this equation is [11]

$$u_n = A\,\mathrm{dn}\big[\beta(n + \delta_1), m\big]e^{-i(\omega t + \delta)}, \tag{9}$$

provided

$$A^2\,\mathrm{cs}^2(\beta,m) = 1, \qquad \beta = \frac{2K(m)}{N_P},$$
$$\omega = -\nu = -2\frac{\mathrm{dn}(\beta,m)}{\mathrm{cn}^2(\beta,m)}. \tag{10}$$

The other solution is

$$u_n = A\sqrt{m}\,\mathrm{cn}\big[\beta(n + \delta_1), m\big]e^{-i(\omega t + \delta)}, \tag{11}$$

provided

$$A^2\,\mathrm{ds}^2(\beta,m) = 1, \qquad \beta = \frac{4K(m)}{N_p},$$
$$\omega = -\nu = -2\frac{\mathrm{cn}(\beta,m)}{\mathrm{dn}^2(\beta,m)}. \tag{12}$$

Remarkably, even a linear superposition of the two is also an exact solution to the saturable DNLS Eq. (8), i.e. it is easily shown using the recently derived identities for the Jacobi elliptic functions [12] that

$$u_n = \left(\frac{A}{2}\,\mathrm{dn}\big[\beta(n + \delta_1), m\big] + \frac{B}{2}\sqrt{m}\,\mathrm{cn}\big[\beta(n + \delta_1), m\big]\right) \times e^{-i(\omega t + \delta)}, \tag{13}$$

is also an exact solution to Eq. (8) provided

$$B = \pm A, \qquad A^2\big[\mathrm{cs}(\beta,m) + \mathrm{ds}(\beta,m)\big]^2 = 4, \qquad \beta = \frac{4K(m)}{N_p},$$
$$\omega = -\nu = -\frac{4}{\mathrm{dn}(\beta,m) + \mathrm{cn}(\beta,m)}. \tag{14}$$