Physics Letters A 377 (2013) 2766–2770

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Pattern recognition minimizes entropy production in a neural network of electrical oscillators

Robert W. Hölzel *, Katharina Krischer

Physik-Department E19a, Technische Universität München, James-Franck-Strasse 1, D-85748 Garching, Germany

ARTICLE INFO

Article history:
Received 25 February 2013
Received in revised form 10 July 2013
Accepted 11 August 2013
Available online 28 August 2013
Communicated by C.R. Doering

Keywords:
Oscillatory network
Weak coupling
Time-dependent coupling
Global coupling

ABSTRACT

We investigate the physical principle driving pattern recognition in a previously introduced Hopfield-like neural network circuit (Hölzel and Krischer, 2011 [13]). Effectively, this system is a network of Kuramoto oscillators with a coupling matrix defined by the Hebbian rule. We calculate the average entropy production $\langle \mathrm{d}S/\mathrm{d}t \rangle$ of all neurons in the network for an arbitrary network state and show that the obtained expression for $\langle \mathrm{d}S/\mathrm{d}t \rangle$ is a potential function for the dynamics of the network. Therefore, pattern recognition in a Hebbian network of Kuramoto oscillators is equivalent to the minimization of entropy production for the implementation at hand. Moreover, it is likely that all Hopfield-like networks implemented as open systems follow this mechanism.

1. Introduction

Networks of phase oscillators with a coupling topology based on the Hebbian learning rule have received considerable attention by the neural network community in the past two decades [1–13]. The appeal of these networks is twofold: On the one hand, oscillatory neurons are known to play a significant role for information processing in biological systems (see for example the review by Wang [14], especially Sections II–IV and the references therein). On the other hand, the dynamics of Hebbian Kuramoto-networks are closely related to the Hopfield model [15] of an autoassociative memory, which has a comparatively simple physical interpretation in the form of the Ising spin model. Hence, the dynamics of large Hebbian Kuramoto-networks are tractable by the methods of statistical physics.

In its most simple form, a Hebbian Kuramoto-network of N phase oscillators is defined by the equations

$$\dot{\varphi}_i = \frac{1}{N}\sum_{j=1}^{N} w_{ij}\sin(\varphi_j - \varphi_i), \qquad w_{ij} = \sum_{k=1}^{M} \xi_i^k \xi_j^k. \tag{1}$$

Here, φ_i is the phase shift of the i-th oscillator with the phase ϑ_i given by $\vartheta_i(t) = \Omega t + \varphi_i(t)$; all oscillators have the same natural frequency Ω. The coupling matrix w_{ij}, representing the synaptic weights connecting the oscillatory neurons, is determined by the Hebbian rule based on the M memorized patterns $\boldsymbol{\xi}^k$, $k = 1 \ldots M$. The $\boldsymbol{\xi}^k$ are binary pattern vectors of length N with entries $\xi_i^k = \pm 1$. Note that in the following, we are using the terms “oscillator” and “oscillatory neuron” synonymously. We use the expression “neuron” because of the mathematical similarity to the neurons in the original Hopfield model, which will become apparent a little further below.

The system (1) is able to recognize a defective binary pattern $\boldsymbol{\xi}$ as one of the $\boldsymbol{\xi}^k$ if the memorized patterns are suitably chosen (the maximal load rate for random patterns is $\alpha = M/N = 0.042$) [6]. Here, defective means that $\boldsymbol{\xi}$ differs from one of the memorized patterns (e.g. $\boldsymbol{\xi}^1$) in a few entries, while it is substantially different from the others (i.e. $\boldsymbol{\xi} \cdot \boldsymbol{\xi}^1 = \mathcal{O}(N)$ and $\boldsymbol{\xi} \cdot \boldsymbol{\xi}^{i \neq 1} = \mathcal{O}(\sqrt{N})$).

For pattern recognition, the network is initialized to $\boldsymbol{\xi}$ by fixing the phase shifts $\varphi_i^{\text{initial}}$ to either 0 or π, such that $\cos(\varphi_i^{\text{initial}}) = \xi_i$, which is an unstable equilibrium of (1). Under the influence of arbitrarily small fluctuations, the system will now evolve towards a final state close to the correctly memorized pattern $\boldsymbol{\xi}^1$, characterized by $\xi_i^1 \cos(\varphi_i^{\text{final}}) > 0$. Note that the final state does not correspond perfectly to $\boldsymbol{\xi}^1$, which would require $\xi_i^1 \cos(\varphi_i^{\text{final}}) = 1$. Still, the network retrieves the binary information without losses. Also note that Eqs. (1) are invariant under a global rotation of all phase shifts, but we described the pattern recognition process in a special frame of reference for simplicity.

There is a potential function $E(\boldsymbol{\varphi})$ of the dynamical system (1), for which $\dot{\varphi}_i = -\partial E/\partial \varphi_i$, given by

$$-E(\boldsymbol{\varphi}) = \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\cos(\varphi_i - \varphi_j). \tag{2}$$

* Corresponding author. Tel.: +498928912554; fax: +498928912530.
E-mail address: robert.hoelzel@mytum.de (R.W. Hölzel).

0375-9601/$ – see front matter
http://dx.doi.org/10.1016/j.physleta.2013.08.022

The dynamics of pattern recognition can also be understood as the tendency of the network to evolve towards a minimum of this potential function. This potential function is identical to the one for the discrete Hopfield model with the two output levels $V^0 = \cos\pi = -1$ and $V^1 = \cos 0 = 1$ (compare Eq. (7) in Hopfield's paper [15]). The main difference in the system at hand is the inclusion of intermediate states, according to (2).

In the rest of the Letter, we will show that for a particular architecture of a Hebbian Kuramoto-network, $E(\boldsymbol{\varphi})$ is proportional to the entropy production of the oscillatory neurons. Thus, successful pattern recognition is equivalent to the minimization of the entropy production of the network.

The architecture we consider in this Letter, which has been recently introduced [13], is briefly described in the next section. Other than in the model above, this implementation of a Hebbian Kuramoto-network uses oscillators with deliberately chosen large differences in frequency. This allows for the adjustment of the synaptic weights of the network by a single experimental parameter that varies in time [8]. The effective dynamics of the phase shifts, however, remain the same.

2. Network circuit

The electrical circuit of the oscillatory network is shown in Fig. 1. It features N autonomously oscillating circuits, where each frequency is unique. Each oscillator fulfills two conditions: First, the individual output signals U_i can be written as the sine of a phase variable ϑ_i with a constant amplitude U_{amp}, i.e. $U_i(t) = U_{\mathrm{amp}} \sin\vartheta_i(t) = U_{\mathrm{amp}} \sin(\Omega_i t + \varphi_i(t))$. Here, small deviations from the original frequency Ω_i of an oscillator are described by the phase shift φ_i. Note that, even as ϑ_i is increasing by 2π for each period, φ_i may be stationary. To maintain this first condition, the oscillator can only react to external stimuli by adjusting its phase. The second condition on each oscillator is that, when it is subjected to an infinitesimal voltage deviation $\mathrm{d}U_i$ caused by the external circuitry, the jump in the voltage is instantly translated into a change in phase $\mathrm{d}\vartheta_i \propto \cos\vartheta_i \mathrm{d}U_i$. In other words, the phase response curve of each oscillator has to be proportional to $\cos\vartheta_i$.

The frequencies Ω_i form a so-called Golomb ruler [16], which means that $\Omega_i - \Omega_j \neq \Omega_k - \Omega_l$ for different pairs i, j and k, l of oscillators; also, for the minimum and maximum frequencies, $\Omega_{\min} > \Omega_{\max}/3$ holds, such that $\Omega_i - \Omega_j \neq \Omega_k + \Omega_l$ for any pair i, j and k, l of oscillators (the reason for this choice of frequencies will become apparent in Section 3). All oscillators are connected to a common circuit node through an individual resistance R_{int}, which provides the necessary degree of freedom for each oscillator. From this common node, an external impedance $Z_{\mathrm{ext}}(t)$ is connected to ground. Unless $Z_{\mathrm{ext}} = 0$, the oscillators are globally coupled via the potential U_{ext} at the common node, which depends on $I_{\mathrm{ext}} = \sum I_{\mathrm{int},i}$. The external impedance consists of a variable resistance $R_{\mathrm{var}}(t) = R_0 + R_{\mathrm{coup}}/N \sum_{i,j} w_{ij} \cos((\Omega_i - \Omega_j)t)$, a serial negative impedance $Z_{\mathrm{ser}} = -R_0$ and a parallel negative impedance $Z_{\mathrm{par}} = -R_{\mathrm{int}}/N$. R_{coup} is a measure for the strength of the coupling – if $R_{\mathrm{coup}} = 0$, then also $Z_{\mathrm{ext}} = 0$. The coefficients w_{ij} are the same as the synaptic weights in Eq. (1).

In the configuration described above, the phase shifts of the electrical oscillators behave effectively as in (1), if the coupling is weak enough (i.e. if Z_{ext} is sufficiently small) [13].

Note that any type of experimental oscillator (an example are van der Pol oscillators [17]) will only approximately fulfill the conditions mentioned above. Also, the negative impedances Z_{par} and Z_{ser} will show some kind of non-ideal behavior in any experimental realization (in particular there will be a certain delay in the output currents with respect to the input voltages, depending on the specific active circuit elements that are used). Both effects will lead to deviations from the dynamics of (1), which can, however, be kept (almost) arbitrarily small with sufficient experimental effort (i.e. using high bandwidth amplifiers to implement the negative impedances together with near-harmonic, frequency stable oscillators that are slow compared to the amplifier bandwidth). Therefore, for the rest of the Letter, we will assume both that the oscillators are perfectly sinusoidal with a phase response curve proportional to $\cos\vartheta$ and that the negative impedances do not introduce any delay. To illustrate that this approach is indeed justified, we present an example experiment in Appendix A, where pattern recognition is not disturbed by the non-ideal behavior of the circuitry.

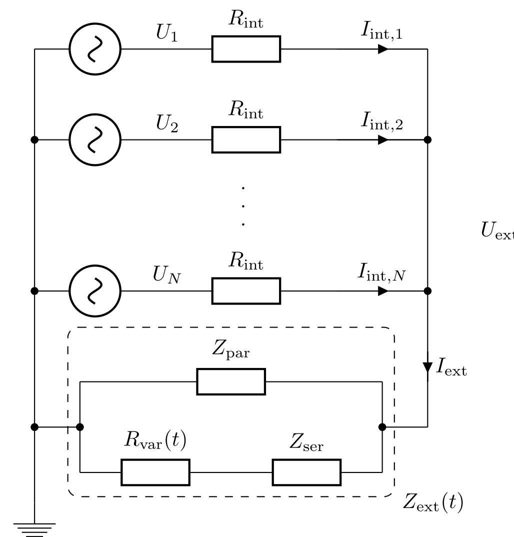

Fig. 1. Circuit diagram of a neural network of sinusoidal electrical oscillators that are coupled globally through an external impedance $Z_{\mathrm{ext}}(t)$. Z_{par} and Z_{ser} are negative impedances, R_{var} is the variable resistance.

3. Entropy production in the circuit

There is an interesting physical interpretation of the dynamical behavior of the phase shifts, other than the fact that it is the solution of the differential equation (1): The system prefers states in which the average combined entropy production $\langle \mathrm{d}S/\mathrm{d}t\rangle_{\mathrm{osc}}$ of all oscillators is minimal. This can also be phrased differently: In a preferred state of the system, the average contribution $\langle P\rangle_{\mathrm{osc}}$ of the oscillators to the overall average power $\langle P\rangle$ dissipated in the network is minimal.

To prove these statements, in the following we will compute the average power output $\langle P\rangle_{\mathrm{osc}}(\boldsymbol{\varphi}) = \langle \mathrm{d}S/\mathrm{d}t\rangle_{\mathrm{osc}}(\boldsymbol{\varphi}) \cdot T$ of all oscillators for any given vector $\boldsymbol{\varphi}$ of phase shifts in the network. Here, T is the ambient temperature and $\boldsymbol{\varphi}$ is the vector with elements $(\varphi_1, \ldots, \varphi_N) = (\vartheta_1 - \Omega_1 t, \ldots, \vartheta_N - \Omega_N t)$, which will *take on a stationary value* after a transient period, due to the fact that (1) has a bounded potential function.

Both the oscillators and the negative impedances supply power to the circuit in Fig. 1. In a stationary state of the network, the dissipated power P is equal to the supplied power:

$$P = P_{\mathrm{osc}} + P_{\mathrm{ext}}^{\mathrm{supp}}, \tag{3}$$

where P_{osc} is the power supplied by all active elements in the oscillators' circuitry and $P_{\mathrm{ext}}^{\mathrm{supp}}$ is the power supplied by all active elements in the circuitry implementing Z_{ser} and Z_{par}. The dissipated power can also be split up according to where the dissipation occurs:

$$P = P_{\mathrm{int}} + P_{\mathrm{ext}}^{\mathrm{diss}}, \tag{4}$$