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Negative probabilities and information gain in weak measurements

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A B S T R A C T

We study the outcomes in a general measurement with postselection, and derive upper bounds for the pointer readings in weak measurement. The probabilities inferred from weak measurements change along with the coupling strength; and the true probabilities can be obtained when the coupling is strong enough. By calculating the information gain of the measuring device about which path the particles pass through, we show that the "negative probabilities" only emerge for cases when the information gain is little due to very weak coupling between the measuring device and the particles. When the coupling strength increases, we can unambiguously determine whether a particle passes through a given path every time, hence the average shifts always represent true probabilities, and the strange "negatives probabilities" disappear.

1. Introduction

Weak measurement, a quantum measurement process with preselection and postselection, was introduced by Aharonov et al. [1]. In a weak measurement, the expectation value of a quantum operator can lay outside the range of the observable's eigenvalues, and this has been confirmed in the field of quantum optics [2]. For very weak interaction between the measuring device and the quantum system, with appropriate initial and final states, the value of the meter's reading can be much larger than that obtained in the traditional quantum measurement, this can be viewed as an amplification effect. This effect has been used to implement high-precision measurements, a tiny spin Hall effect of light has been observed by Hosten and Kwiat [3]; small transverse deflections and frequency changes of optical beams have been amplified significantly [4]. Because of its importance in applications, there has been much work on weak measurement [5–21].

Besides its usefulness in measuring small signals, weak measurement is also used extensively to analyze the foundational questions of quantum mechanics. Weak measurement provides a new perspective to the famous Hardy's paradox [22], and the predictions by Aharonov et al. [23] have been realized in experiments [24]. Using the idea of weak measurement, Lundeen et al. [27] have directly measured the transverse spatial quantum wave function of photons, and Kocsis et al. [28] have observed the average trajectories of single photons in a two-slit interferometer which could not be accomplished in traditional quantum measurements. As commented by Cho [29], weird weak measurements are opening new vistas in quantum physics.

In this Letter we study the outcomes of the pointer readings and derive the upper bounds in a weak measurement, we apply weak measurement to analyze Hardy's paradox and discuss when the "negative probabilities" (observed in [24]) emerge. Just as negative kinetic energy [25] and superluminal group velocities [26], observable negative probabilities seem confusing. In fact, the "negative probabilities" in Hardy's gedanken experiment are not true probabilities. The "negative probabilities" just indicate that the pointer's average shifts has an opposite sign from what is expected with the presence of positive number of particles, hence the "negative probabilities" just indicate a negative effect, actually. In the literature [30], it has been obtained that the effect of signal amplification via weak measurement only exist for the cases when the interaction between the measuring device and the quantum system is very weak. Do the "negative probabilities" only exist in the case of very weak interactions, just as does the amplification effect? How can one view the emergence of the "negative probabilities" from an information theoretical perspective? We shall discuss these questions in this Letter.

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2. The range of the pointer's shifts in weak measurement

To perform a weak measurement of an observable $\mathbf{A}$, we need four steps. First, we prepare the quantum systems to be measured in the initial state $|\psi_i\rangle$. Second, let the quantum systems interact weakly with a measuring device. Third, we perform a strong measurement and select the quantum systems in the final state $|\psi_f\rangle$. Finally, we record the readings of the measuring device conditioned on successfully obtaining the final state $|\psi_f\rangle$ of the system. The weak value was introduced by Aharonov et al. [1]

$$\mathbf{A}_w = \frac{\langle\psi_f|\mathbf{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}, \tag{1}$$

which can be written as $\mathbf{A}_w = a + ib$ (with $a, b \in \mathcal{R}$). The interaction Hamiltonian is generally modeled as

$$H = g\delta(t-t_0)\mathbf{A}\otimes p, \tag{2}$$

where g is the coupling strength with $g \geqslant 0$ and p is the pointer momentum conjugate to the position coordinate q. We assume that A is dimensionless and we use the natural unit $\hbar = 1$. Jozsa [6] has given the final average shifts of pointer position and momentum

$$\begin{aligned}\delta q &= \langle q\rangle' - \langle q\rangle = ga + gb\cdot\big\langle\{p,q\}\big\rangle,\\ \delta p &= \langle p\rangle' - \langle p\rangle = 2gb\cdot \mathrm{Var}_p\,.\end{aligned} \tag{3}$$

Here $\langle\hat{o}\rangle$ denotes the expectation value of an observable $\hat{o}$ of the device in its initial state, and $\langle\hat{o}\rangle'$ with a prime denotes the corresponding value in the final state of the device after the interaction and postselection. $\mathrm{Var}_p = (\Delta p)^2$ ($\mathrm{Var}_q = (\Delta q)^2$) denotes the variance of the pointer momentum (position) in the initial pointer state, and $\{p,q\} = pq + qp$ denotes the anti-commutator.

When one chooses appropriate initial and final states of the system such that $\langle\psi_f|\psi_i\rangle \to 0$, both the real and imaginary parts of the weak values can become arbitrarily large, and one might think that the average shifts of the pointer's position and momentum could become arbitrarily large as well, according to Eq. (3). However, in order to obtain Eq. (3), approximations are used and only the first-order terms of g are kept; the approximations as well as Eq. (3) are no longer valid when $\langle\psi_f|\psi_i\rangle \to 0$. It was pointed out in [31] that the average pointer shifts may have an upper bound, and this observation was also confirmed in [30,32]. For the case when a qubit system weakly interacts with a pointer that was initially in a Gaussian state, it is shown in [30] that the maximum average pointer shift δq (δp) over all possible pre- and post-selections (PPS) are bounded from above by the standard deviation Δq (Δp) of the pointer variable in the initial state, i.e., $\max\{\delta q\} \leqslant \Delta q$ and $\max\{\delta p\} \leqslant \Delta p$. In the following, we shall show that these upper bounds still hold for the more general cases.

Wu and Li proposed a more general and precise framework of weak measurement by retaining the second-order terms of the coupling strength g [31]. When the initial pointer state ρ_d satisfies $\langle p\rangle = 0$ and $\langle q\rangle = 0$ (these conditions can be always satisfied by choosing a suitable "zero point") and the variance of p is not changing with time, the expressions of the average shifts in q and p are obtained as

$$\delta q = \frac{g\,\mathrm{Re}\,\langle\mathbf{A}\rangle_w}{1 + g^2\,\mathrm{Var}_p(\langle\mathbf{A}\rangle_w^{1,1} - \mathrm{Re}\,\langle\mathbf{A}^2\rangle_w)}, \tag{4}$$

$$\delta p = \frac{2g\,\mathrm{Im}\,\langle\mathbf{A}\rangle_w\,\mathrm{Var}_p}{1 + g^2\,\mathrm{Var}_p(\langle\mathbf{A}\rangle_w^{1,1} - \mathrm{Re}\,\langle\mathbf{A}^2\rangle_w)}, \tag{5}$$

where

$$\langle\mathbf{A}\rangle_w = \frac{\mathrm{tr}(\Pi_f\mathbf{A}\rho_s)}{\mathrm{tr}(\Pi_f\rho_s)}, \qquad \langle\mathbf{A}\rangle_w^{1,1} = \frac{\mathrm{tr}(\Pi_f\mathbf{A}\rho_s\mathbf{A})}{\mathrm{tr}(\Pi_f\rho_s)}, \tag{6}$$

here ρ_s is the initial state of the system (preselection), and Π_f is a general postselection that could be a projection onto a final pure state or a subspace.

When the coupling strength is very weak, i.e., $g\Delta p \ll 1$, we search for the maximum shifts of the measuring device using the expressions in Eqs. (4) and (5). The absolute value of the shift δq

$$|\delta q| \leqslant \frac{g|\langle\mathbf{A}\rangle_w|}{|1 + (g\Delta p)^2(\langle\mathbf{A}\rangle_w^{1,1} - \mathrm{Re}\,\langle\mathbf{A}^2\rangle_w)|}. \tag{7}$$

If the observable $\mathbf{A}$ is a projective operator which satisfies $\mathbf{A}^2 = \mathbf{A}$, we have

$$|\delta q| \leqslant \frac{g|\langle\mathbf{A}\rangle_w|}{|1 + (g\Delta p)^2(\langle\mathbf{A}\rangle_w^{1,1} - \mathrm{Re}\,\mathbf{A}_w)|}. \tag{8}$$

First we prove $\langle\mathbf{A}\rangle_w^{1,1} \geqslant |\langle A\rangle_w|^2$. Let $C = \langle\mathbf{A}\rangle_w^{1,1} - |\langle A\rangle_w|^2$, we have

$$C = \frac{\mathrm{tr}(\Pi_f\mathbf{A}\rho_s\mathbf{A})\,\mathrm{tr}(\Pi_f\rho_s) - \mathrm{tr}(\Pi_f\mathbf{A}\rho_s)\,\mathrm{tr}(\rho_s\mathbf{A}\Pi_f)}{(\mathrm{tr}(\Pi_f\rho_s))^2}. \tag{9}$$

The spectral decomposition of the operators ρ_s and Π_f can be written as

$$\rho_s = \sum_i p_i|\psi_i\rangle\langle\psi_i|, \qquad \Pi_f = \sum_j q_j|\phi_j\rangle\langle\phi_j|, \tag{10}$$

where $\sum_i p_i = 1$, $p_i \geqslant 0$, and $q_j = 0$ or 1 since Π_f is a projective operator. The numerator of Eq. (9) is

$$\begin{aligned}F = {}&\bigg(\sum_{ij} p_i q_j\big|\langle\psi_i|\mathbf{A}|\phi_j\rangle\big|^2\bigg)\bigg(\sum_{mk} p_m q_k\big|\langle\psi_m|\phi_k\rangle\big|^2\bigg)\\ &- \bigg|\bigg(\sum_{ij} p_i q_j\langle\psi_i|\mathbf{A}|\phi_j\rangle\langle\phi_j|\psi_i\rangle\bigg)\bigg|^2.\end{aligned} \tag{11}$$

We construct two vectors

$$\begin{aligned}|a\rangle &= \sum_{ij}\sqrt{p_i q_j}\langle\psi_i|\mathbf{A}|\phi_j\rangle|i,j\rangle,\\ |b\rangle &= \sum_{ij}\sqrt{p_i q_j}\langle\psi_i|\phi_j\rangle|i,j\rangle,\end{aligned} \tag{12}$$

where $\{|i,j\rangle\}$ is a orthonormal basis satisfying $\langle i,j|i',j'\rangle = \delta_{ii'}\delta_{jj'}$. So Eq. (11) can be rewritten as

$$F = \langle a|a\rangle\langle b|b\rangle - \big|\langle a|b\rangle\big|^2. \tag{13}$$

From Schwarz inequality we have $F \geqslant 0$, equality holds when $|a\rangle$ is proportional to $|b\rangle$. The denominator of Eq. (9) is positive, so we have

$$\langle\mathbf{A}\rangle_w^{1,1} \geqslant \big|\langle A\rangle_w\big|^2. \tag{14}$$

In particular, when PPS are all pure states, equality holds.

As $g\Delta p \leqslant 1$, let $K = 1 + (g\Delta p)^2(\langle\mathbf{A}\rangle_w^{1,1} - \mathrm{Re}\,\mathbf{A}_w)$, and from Eq. (14) we have

$$\begin{aligned}K &\geqslant 1 + (g\Delta p)^2\big(|\mathbf{A}_w|^2 - |\mathbf{A}_w|\big)\\ &\geqslant \bigg(1 - \frac{1}{2}g\Delta p|\mathbf{A}_w|\bigg)^2 + \frac{3}{4}\big(g\Delta p|\mathbf{A}_w|\big)^2\\ &\geqslant 0.\end{aligned} \tag{15}$$

From Eqs. (8) and (15), we obtain