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Stresses and strains in a deformable fractal medium and in its fractal continuum model

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ABSTRACT

The model of fractal continuum accounting the topological, metric, and dynamic properties of deformable physical fractal medium is suggested. The kinematics of fractal continuum deformation is developed. The corresponding geometric interpretations are provided. The concept of stresses in the fractal continuum is defined. The conservation of linear and angular momentums is established. The mapping of mechanical problems for physical fractal media into the corresponding problems for fractal continuum is discussed.

1. Introduction

Most natural and engineering materials are inherently heterogeneous due to the presence of microstructure [1]. In the past two decades it was recognized that microstructures of real heterogeneous materials frequently possess formidably complicated architecture characterized by statistical scale invariance over many length scales [1–7]. For such materials the classical approximation of homogeneous Euclidean continuum is inapplicable, because the heterogeneities play an important role on almost all scales. At the same time, the fractal geometry offers helpful scaling concepts to characterize and model the scale invariant structures of heterogeneous media [1–30].

While there is no canonical definition of fractals, mathematically a fractal is commonly viewed as an object the metric dimension of which D (e.g., Hausdorff, Minkowski, self-similarity, etc.) is larger than its topological dimension d [2], except of special cases such as the Hilbert space-filling curves with $D = d$ [31]. It is obvious that fractals with $d \leqslant D < n$ cannot continuously fill the embedding Euclidean space E^n. Consequently, the properties of fractal structures are essentially discontinuous non-differentiable functions of the Euclidean coordinates in E^n [32]. Accordingly, to deal with fractal materials, it was suggested the concept of fractal continuum [33] the overall properties of which are defined as the analytic envelopes of non-analytic functions characterizing the fractal $\Phi_n^D \subset E^n$ under study [32–45]. In this way, the fractal continuum $\Phi_D^n \subset E^n$ can be defined as n-dimensional region of E^n equipped with appropriate fractional metric, measure, and vector differential calculus, such that its properties (density, displacements, velocities, etc.) are describable by the continuous (or, at worst, piecewise continuous) differentiable functions of space and time variables [42,43]. It should be emphasized that in contrast to fractals, the topological dimension of fractal continuum $\Phi_D^n \subset E^n$ is, per definition, equal to the dimension of the embedding Euclidean space, that is $d_{FC} = n > D$. This immediately implies that the density of admissible states in $\Phi_D^n \subset E^n$ should be scale dependent [44,45].

Although the measure of fractal continuum can be fixed by a natural requirement that the mass of any region $W \subset \Phi_D^3$ of characteristic size L should display the same scaling behavior as the fractal medium under study, that is $M(W) \propto L^D$ [33], there are very different ways to define the metric and fractional calculus in the fractal continuum. Accordingly, quite different models of fractal continua were suggested to aboard the problems of mechanics and electrodynamics in fractal media [33–46]. These different models lead to quite different solutions of the same problems for the modeled fractal medium. Nonetheless that the strength of fractal continuum models can be ranked by its abilities to explain and predict the results of experimental studies, the mathematical and physical self-consistency of the model should be assured before its applications to a specific problem.

The mechanics of deformable medium cannot be deduced from the laws of mechanics of material points and rigid bodies. Hence, additional assumptions are needed to introduced, such that new notions of internal and external forces, stresses, and the equilibrium equation should emerge. In this context, the mechanical behavior of fractal media has topological and geometrical aspects which should be accounted within the fractal continuum framework. The fractal (mass or metric) dimension D characterizes how

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the extensive (e.g. mass) and intensive (e.g. density, $\rho \propto L^{n-D}$) properties of heterogeneous medium scale with system size in E^3, but it tells us nothing about the connectivity and topological properties of the fractal, such that fractals of different topology and connectivity can have the same mass (metric) dimension [11]. Therefore, to account the fractal topology of medium one needs to endow the fractal continuum model $\Phi_D^n \subset E^n$ with additional appropriate dimensional numbers.

In this Letter, we suggest the model of deformable fractal continuum accounting the topology and metric of fractal material. The model is used to develop the fractal continuum mechanics of heterogeneous materials with scale-invariant (micro-)structures.

2. Fractal continuum ${}_{\alpha}^{d_s}\Phi_D^3 \subset E^3$

Although, in mathematics, fractals can be defined without any reference to the embedding space [47], in real life fractal materials reside in the three-dimensional space and occupy a well-defined volume V_3 in E^3. Accordingly, the scaling properties of fractal pattern $\Phi_3^D \subset E^3$ can be characterized by a set of fractional dimensionalities [16]. Most definitions of dimension numbers are based on the concept of fractal covering by balls (cubes, tubes, etc.) of some size ε, or at most ε. In mathematics these covers are considered in the limit $\varepsilon \to 0$. At the same time, it was noted that, in many cases, the number of n-dimensional coats need to cover the mathematical fractal of linear size L in E^3 scales as $N \propto (L/\varepsilon)^D$. It is precisely this power-law behavior gives rise to use the powerful tools of fractal geometry to deal with physical patterns $\Phi_3^D \subset E^3$ exhibiting statistical scale invariance only within a wide, but bounded interval of length scale $\xi_0 < \varepsilon \leqslant L < \xi_C$, where ξ_0 and ξ_C are the lower and upper cut-offs of the physical origin [48]. Hence, strictly speaking, physical fractals are closer to the concept of pre-fractals obtained after finite number of iterations, whereas the true fractal can be obtained in the limit of infinite number of iterations ($\varepsilon \to 0$ while $\xi_0 = 0$).

To model the fractal medium within a continuum framework, in this work we define three-dimensional fractal continuum ${}_{\alpha}^{d_s}\Phi_D^3 \subset E^3$ as three-dimensional region of the embedding Euclidean space E^3 filled with continuous matter and endowed with appropriate fractional measure, metric, and norm, as well as a set of rules for integro-differential calculus and a proper Laplacian accounting the metric, connectivity and topological properties of the modeled fractal medium.

2.1. The measure of fractal continuum

Since the fractal continuum has topological dimension $d_{FC} = 3 > D$ we can use the conventional rules of Lebesgue integration in E^3, whereas the fractal measure can be accounted via the definition of scale dependent density of admissible states $c_3(x)$ in $\Phi_D^3 \subset E^3$. In this way, the mass of any cubic (or spherical) region $W \subset \Phi_D^3$ of the characteristic size L is assumed to scale as

$$M(W) = \int_{W\in\Phi_3^D} \rho_0\, dV_F = \int_{W\in E^3} \rho_c c_3(x_i)\, dV_3\big(x_i \in E^3\big) = \int_{W\in\Phi_D^3} \rho_c\, dV_D = m_0\left(\frac{L}{\xi_0}\right)^D, \tag{1}$$

where dV_F, dV_3 and dV_D are the infinitesimal volume elements in the fractal medium Φ_3^D, Euclidean space E^3, and fractal continuum Φ_D^3, respectively, x_i are the Cartesian coordinates in E^3 (see Fig. 1), ξ_0 is the characteristic size of elemental Euclidean components of mass m_0 and mass density ρ_0 from which the physical fractal medium is made, ρ_c is the density of fractal continuum (for example, in the case of porous fractal medium $\rho_c = (1-\phi)\rho_0$, while ϕ is the total porosity). Notice that $\xi_0 > 0$ accounts the pre-fractal nature of physical fractal medium and so the corresponding fractal continuum should obey the scaling property (1) for $L > \xi_0$ only.

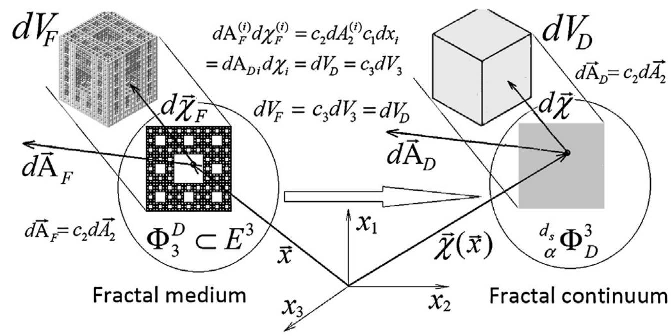

Fig. 1. Mapping of essentially discontinuous Menger sponge $\Phi_3^D \subset E^3$ ($d_f = d_\ell = D = \ln 20/\ln 3$) into the fractal continuum ${}_{\alpha}^{d_s}\Phi_D^3 \subset E^3$ with $D = \ln 20/\ln 3$, $D_S = \ln 8/\ln 3$, and $\zeta = \ln 2.5/\ln 3 < \alpha = d_\ell/3 = \ln 20/\ln 27$.

To account the topological properties of fractal medium, we noted that the connectivity and topology of the fractal $\Phi_3^D \subset E^3$ can be specified by the chemical dimension d_ℓ and the fractal dimensions $D_S^{(i)}$ of intersections between the fractal and the Cartesian planes in the embedding Euclidean space E^3 [42]. The chemical dimension quantifies how the "elementary" structural units of the (pre-)fractal structure are "glued" together to form the entire fractal object [16]. So, d_ℓ tells us "how many directions" the observer feels in the configuration space by making *static* measurements. Therefore, d_ℓ determines the minimal number of independent coordinates needs to define the point position in the fractal, in the same way as the topological dimension d determines the number of independent coordinates (e.g. the Cartesian coordinates) in the Euclidean manifold. Hence, the number of mutually orthogonal independent coordinates which can be defined in the fractal with $d_\ell < 3$ is less than 3. Although one can speak about the fractional number of coordinates [49], in a fractal with $2 \leqslant d_\ell < 3$ it is more convenient to define two fractional coordinates $\chi_i, \mathrm{A}_i \in \Phi_D^3$, such that the infinitesimal volume elements in $\Phi_D^3 \subset E^3$ can be decomposed as

$$dV_D = d\chi_i(x_i)\, d\mathrm{A}_i(x_{j\neq i}), \tag{2}$$

where $d\mathrm{A}_i$ is the infinitesimal area element on the intersection between $\Phi_D^3 \subset E^3$ and the Cartesian plane $(x_j, x_k) \in E^3$ having the fractal dimension $D_S^{(i)}$, while $d\chi_i$ is the infinitesimal length element along the lines parallel to the Cartesian axis $i \neq j,k$ normal to the Cartesian plane (x_j, x_k). Accordingly, dV_D can be decomposed

$$dV_D = d\chi_\zeta\, d\mathrm{A}_D = c_1^{(1)}\, dx_1\, c_2^{(1)}\, dA_2^{(1)} = c_1^{(2)}\, dx_2\, c_2^{(2)}\, dA_2^{(2)} = c_1^{(3)}\, dx_3\, c_2^{(3)}\, dA_2^{(3)} = c_3\, dV_3 \tag{3}$$

where $dA_2^{(i)}$ denote the infinitesimal area elements on two-dimensional Cartesian planes and $c_2^{(i)}(x_{j\neq i})$ is the density of admissible states on the intersection between Φ_D^3 and the Cartesian plane normal to the i-axis, while $c_1^{(i)}(x_i)$ is the density of admissible states along lines parallel to the i-axis (see Fig. 1). From Eq. (3) immediately follows that

$$c_3(x_k) = c_1^{(i)}(x_i)\, c_2^{(i)}(x_{j\neq i}), \tag{4}$$

but the functional form of $c_3(x_k)$ can be defined in the unique way as $c_3 = c_1^{(1)}(x_1)c_1^{(2)}(x_2)c_1^{(3)}(x_3)$, if and only if $d_\ell = 3$, such