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Observation of rogue wave triplets in water waves

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ABSTRACT

Doubly-localised breather solutions of the nonlinear Schrödinger equation (NLS) are considered to be appropriate models to describe rogue waves in water waves as well as in other nonlinear dispersive media such as fibre optics. Within the hierarchy of this type of formations, the Peregrine breather (PB) is the lowest-order rational solution. Higher-order solutions of this kind may be understood as a nonlinear superposition of fundamental Peregrine solutions. These superpositions are nontrivial and admit only a fixed well prescribed number of elementary breathers in each higher-order solution. Here, we report first observation of second-order solution which in reality is a triplet of rogue waves.

Rogue waves is the subject of intense discussions in the water wave community [1–4] as well as in other fields of physics. Well accepted physical explanation for the sudden formation of high amplitude waves on the surface of water is modulational instability (MI) [5,6]. In the approximation of deep-water waves, the MI provides a condition for extreme localisation both in time and space thus creating the waves which appear from nowhere and disappear without a trace. This happens due to the instability that starts from a regular nonlinear Stokes wave train. In the narrow spectrum approximation, the MI can be studied within the framework of the focusing NLS [7,8]. This is a weakly-nonlinear evolution equation describing the propagation in time and space of wave trains in deep-water. In scaled form, it can be written as:

$$i\psi_T + \psi_{XX} + 2|\psi|^2\psi = 0, \tag{1}$$

where ψ is the complex amplitude, T is time in the frame moving with the group velocity and X is the normalised spatial coordinate. Note that the NLS used here is different from the one in [9]. Namely, the T-variable is multiplied by 2. Thus, the solution given below has different scaling along the T-axis.

The MI phenomenon can be described using the exact solutions of the NLS [1]. Recent experiments on the PB [10] which is an exact solution of the NLS localised in space and time have been observed in several media [11–14]. These experiments confirmed the validity of breather dynamics in optics, in water waves and in plasma, respectively. Higher-order solutions are also of great interest as their effect can be much more powerful than of a single PB.

Generally, the doubly-localised formations comprise a hierarchy of solutions of increasing order with the lowest-order one being the PB. The higher-order solutions contain several PBs. The interesting point is that the number or PBs M do not coincide with the order of the solution n. Their number in the superposition is higher than n and can be expressed as $M = n(n+1)/2$. Thus, for the second-order solution the number of PBs is 3, for the third-order solution, the number of PBs is 6, etc. One of the main conclusions from this result is that there is no solution with nonlinear superposition of two PBs. If higher than one, their number has to be 3 or 6. In this work, we demonstrate, experimentally, the existence of rogue wave triplets with 3 PBs in it.

Another remarkable property of higher-order rogue wave solutions is that they are arranged in special geometrical forms. PBs in triplets are strictly located at the apexes of the equilateral triangle in the $(X, 2T)$-plane. The size and orientation of the triangle are controlled by two real parameters of the solution. Thus, when observing the solution either in space or time, we can see the PBs at special distances from each other.

To be specific, each exact expression for the hierarchy of doubly-localised rational solutions can be written in terms of polynomials $G_n(X,T)$, $H_n(X,T)$ and $D_n(X,T)$:

$$\psi_n(X,T) = \left((-1)^n + \frac{G_n(X,T) + iH_n(X,T)}{D_n(X,T)}\right)\exp(2iT), \tag{2}$$

where $n \in \mathbb{N}$ denotes the order of the solution. It is important to point out that all these solutions $\psi_n(X,T)$ tend to the scaled uniform second-order Stokes wave $\exp(2iT)$ as the spatial X and the time T coordinates tend to infinity. In the case of the lowest first-order solution ($n = 1$) represented by the PB [10], these polynomials are defined as

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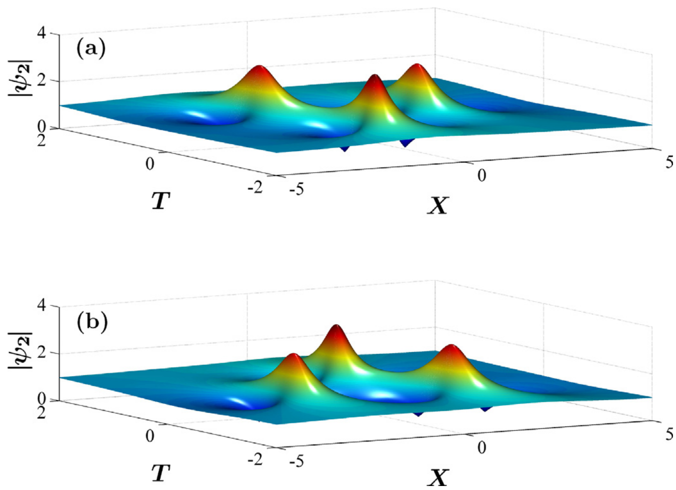

Fig. 1. Rogue wave triplet solution of the NLS ψ_2 (a) for $\beta = 50$, $\gamma = -50$ and (b) for $\beta = -50$, $\gamma = 50$.

$$G_1 = 4, \qquad H_1 = 16T \quad \text{and} \quad D_1(X,T) = 1 + 4X^2 + 16T^2.$$

If we ignore the trivial scaling parameter and translations along X and T variables, the solution is fixed.

On the other hand, the second-order solution ψ_2 that describes rogue wave triplet dynamics [9] comprises a two-parameter family. It is defined by:

$$\begin{aligned} G_2 = {} & 12\big[3 - 16X^4 - 24X^2\big(16T^2 + 1\big) \\ & - 4\beta X - 80(2T)^4 - 72(2T)^2 + 8\gamma T\big], \end{aligned} \tag{3}$$

$$\begin{aligned} H_2 = {} & 24\big[2T\big(15 - 16X^4 + 24X^2 - 4\beta X\big) \\ & - 8\big(4X^2 + 1\big)(2T)^3 - 16(2T)^5\big] \\ & + 24\big(\gamma\big(8T^2 - 2X^2 - 0.5\big)\big), \end{aligned} \tag{4}$$

$$\begin{aligned} D_2 = {} & 64X^6 + 48X^4\big(16T^2 + 1\big) + 12X^2\big(3 - 16T^2\big)^2 \\ & + 64(2T)^6 + 432(2T)^4 + 396(2T)^2 + 9 \\ & + \beta\big[\beta + 4X\big(12(2T)^2 - 4X^2 + 3\big)\big] \\ & + \gamma\big[\gamma + 8T\big(12X^2 - 16T^2 - 9\big)\big], \end{aligned} \tag{5}$$

where the two free real parameters $\beta, \gamma \in \mathbb{R}$ describe the orientation and the size of the triplet on the $(X, 2T)$-plane. As mentioned above, the first-order components of the triplet form an equilateral triangle in this plane centred at the origin. Each side of this triangle is given by an approximate expression $\approx \sqrt{3}(\beta^2 + \gamma^2)^{1/6}/2$ which provides an estimate of the size of the whole structure in space–time. When $\beta = 0$, the maximum of one of the components is located on the T-axis either on positive or negative side of it depending on the sign of γ. Two other components then appear at the other half-plane with maxima located at equal T-values. If $\beta \neq 0$, the structure rotates around the origin by $\theta = -\frac{1}{3}\arctan(\frac{\beta}{\gamma})$ relative to the above position. In particular case, when $\beta = -\gamma$, the angle of rotation is $\pi/12$, i.e. 15^0. In this case, the three components are separated both in time and in space. We should keep in mind that only positions of the components rotate. The orientation of each individual component on $(X, 2T)$-plane remains unchanged. Initial conditions in the experiment were defined by the above equations with large negative X and the values of $\beta = -\gamma$ sufficient for their complete separation. The value of $|X|$ is basically limited by the length of the tank.

Fig. 1 shows two examples of a rogue wave triplet when the parameters are chosen to be $\beta = 50$, $\gamma = -50$ and $\beta = -50$, $\gamma = 50$, respectively. The two choices provide the same size of the triangle but correspond to different orientations. This can be clearly seen from Fig. 1. When the two parameters are zero, $\beta = 0$ and $\gamma = 0$, the solution focuses at the origin with strong localisation at $X = 0$ and $T = 0$ [15] which amplifies the carrier by a remarkably high factor of five. This solution (dubbed super-rogue wave) has been observed up to now only in water waves [16,4]. Similar higher-order cluster solutions are described in [17–19].

In order to perform the experiments, the two-parameter-dependent second-order rational solution has to be written in dimensional units in order to satisfy the dimensional deep-water wave NLS [8]:

$$i\left(\frac{\partial A}{\partial t} + \frac{\omega_0}{2k_0}\frac{\partial A}{\partial x}\right) - \frac{\omega_0}{8k_0^2}\frac{\partial^2 A}{\partial x^2} - \frac{\omega_0 k_0^2}{2}|A|^2 A = 0. \tag{6}$$

For this purpose we apply the transformations [1]:

$$T = -\frac{\omega_0}{8k_0^2}t, \qquad X = x - \frac{\omega_0}{2k_0}t, \qquad \psi = \sqrt{2}k_0^2 A, \tag{7}$$

to Eq. (1). The wave frequency ω_0 and the wavenumber k_0 are connected by the dispersion relation $\omega_0 = \sqrt{gk_0}$, where $g = 9.81\ \mathrm{m\,s^{-2}}$ denotes the gravitational acceleration.

To second order in steepness, the surface elevation $\eta(x,t)$ is related to the breather solution $A(x,t)$ as follows:

$$\begin{aligned} \eta(x,t) = {} & \mathrm{Re}\big(A(x,t)\exp\big[i(k_0 x - \omega_0 t)\big]\big) \\ & + \frac{1}{2}k_0 A^2(x,t)\exp\big[2i(k_0 x - \omega_0 t)\big]. \end{aligned} \tag{8}$$

This formula is used to determine the initial flap motion as well as for comparing the experimental results with theory. Experiments have been conducted in the facility, described earlier in [16]. In order to determine the initial condition for the triplet solution, provided by theory, it is sufficient to fix three parameters in the surface elevation expression (8), related to the second-order solution in dimensional units, i.e. $A(x,t) = \psi_2(x,t)$. These are the amplitude a_0, the steepness $\varepsilon_0 := a_0 k_0$ of carrier and the starting spatial co-ordinate, x_{initial}, of the wave evolution.

The single-flap motion at the starting side of the tank generates exact initial condition provided by the NLS theory. In particular, we used the breather triplet solution given by Eqs. (3)–(5). The computerised equipment takes into account the response function of the wave generating paddle which is described in [20]. The major difficulty of the experiment with the second-order triplet solution is that it requires long propagation distances. These are longer than the actual length of the tank which is a significant limitation. To overcome this difficulty the experiment can be done in sequences. Namely, the tank's length can be effectively increased by recording a signal measured at a specific position from the paddle in the first part of the sequence and regenerating the measured wave profile in the second part of the sequence. Repeating this procedure with sufficiently high accuracy, the flume's length can be extended several times. To ensure the accuracy, it is crucial to avoid wave reflections. The measured signals should be undisturbed as much as possible. In order to reach the required high accuracy, the measuring wave gauge is placed 9 m from the single paddle being separated by 3 m from the installed beach. At this distance, the reflections are barely noticeable for the chosen carrier parameters.

The first set of experiments has been conducted for a carrier amplitude of $a_0 = 0.5$ cm and a steepness value of $\varepsilon_0 = 0.08$, thus, the wavenumber and wave frequency are $k_0 = 16\ \mathrm{rad\,m^{-1}}$ and $\omega_0 = 12.52\ \mathrm{rad\,s^{-1}}$, respectively. The parameter values of the triplet are $\beta = 50$ and $\gamma = -50$ while the initial co-ordinate is $x_{\mathrm{initial}} = -30$ m. These parameters have been chosen in order to start the experiments with small amplitude amplification and to avoid wave breaking, which may result from the extreme steepening of the waves during the evolution [21]. Fig. 2 shows the