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Correlations between quantum and stochastic systems with a dephasing coupling

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ABSTRACT

Correlations between single qubit and classical environment are studied by means of the stochastic Liouville equation, where a dephasing coupling between them is assumed. When the dephasing of the qubit is characterized by the two-state-jump Markov process, the properties of the total, classical and quantum correlations are examined.

1. Introduction

A quantum system interacting with a surrounding environment undergoes irreversible time evolution which is caused by correlation between the quantum system and environment [1–3]. When an environmental system is modeled quantum mechanically, correlation between two quantum systems is obtained. Such correlation is classified into total, quantum and classical correlations [4–8]. In particular, quantum correlation is further classified into entanglement [9] and quantum discord [5]. The latter is quantum correlation existing in a separable state. Tracing out the environmental degrees of freedom from a total state, we can obtain a reduced state of a relevant quantum system [1,3]. Besides a quantum mechanical environment, a classical environment is also useful for investigating decoherence of a quantum system [10–22]. Such an environment can be modeled by means of a classical stochastic process [3,23]. In general, a bipartite system whose density matrix is given by $\hat{\rho} = \sum_k p_k \hat{\rho}_k \otimes |k\rangle\langle k|$ is a quantum-classical system, where $|k\rangle$ is an orthogonal vector of one part of the total system, $p_k \geqslant 0$ and $\sum_k p_k = 1$. In this Letter, we consider the case that influences of a classical environment on a quantum system are described by means of a stochastic process. In the rest of this Letter, we refer to such an environment as a stochastic environment. Although a state of quantum system and stochastic environment can be written in the form of $\hat{\rho} = \sum_k p_k \hat{\rho}_k \otimes |k\rangle\langle k|$, the time evolution is different from that of quantum system and non-stochastic environment. In fact, time evolution of a non-stochastic bipartite system obeys the ordinary Liouville–von Neumann equation $\partial\hat{\rho}(t)/\partial t = -(i/\hbar)[\hat{H}_{\text{total}}, \hat{\rho}(t)]$ [1–3], where $\hat{H}_{\text{total}}$ is a bipartite Hamiltonian, while time evolution of quantum and stochastic systems is determined by the stochastic Liouville equation [24–29] which provides non-unitary time evolution of the total system. Correlation between quantum and stochastic environment is not so clear, though many works use stochastic processes for investigating decoherence of a quantum system [10–22]. The previous works have paid attention to the decoherence of a relevant quantum system under the influence of stochastic fluctuations. Hence it will be worth investigating correlation between quantum system and stochastic environment.

The methods for investigating a quantum system interacting with a stochastic environment are classified into two methods [24,25]. One uses a stochastic Hamiltonian [24], where the effect of an environment on a quantum system is included as a fluctuating classical field described by means of a stochastic process. The model that a quantum system is influenced by a stochastic fluctuation has been initiated by Kubo [30] and Anderson [31] to investigate the spin relaxation process. So such a model is sometimes called the Kubo–Anderson model. In this model, time evolution of a density matrix $\hat{\rho}(t)$ is determined by the stochastic Liouville equation $\partial\hat{\rho}(t)/\partial t = -(i/\hbar)[\hat{H}(t), \hat{\rho}(t)]$ with a stochastic Hamiltonian $\hat{H}(t)$ [24]. The relevant quantum system is described by the density matrix derived by taking the average of $\hat{\rho}(t)$ over the stochastic process. The other method uses the different type of the stochastic Liouville equation (see Section 2) which has also been developed by Kubo to investigate the relaxation phenomena [25].

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In this case, the stochastic Liouville equation determines time evolution of a compound state of a whole system consisting of quantum system and stochastic environment, the marginals of which are respectively a reduced density matrix of the quantum system and a probability distribution of a stochastic variable. Although both methods provide the same results for time evolution of a relevant quantum system [29], time evolution of correlation between quantum system and stochastic environment is hardly studied by means of the former method [24]. Hence to investigate correlations between them, we need to use the stochastic Liouville equation in the latter method [25]. In this Letter, using the stochastic Liouville equation [25,27,29], we will derive correlations between quantum system and stochastic environment, where the quantum system is a single qubit (a two-level quantum system) [9] and the fluctuation of the quantum system caused by the environment is characterized by means of the two-state-jump Markov process [23]. We will examine whether the obtained correlations are quantum mechanical or classical. Furthermore, we will investigate the time evolution of correlations in the cases of the slow and fast modulations of the dephasing.

This Letter is organized as follows. In Section 2, we briefly explain the stochastic Liouville equation for a compound state of a whole system consisting of single qubit and stochastic environment, where the phase fluctuation of the qubit caused by the environment is described by means of a stochastic variable of the two-state-jump Markov process, and we derive an exact solution of the stochastic Liouville equation. We obtain a conditional density matrix of the qubit that is essential for deriving correlations between the qubit and the stochastic environment. In Section 3, we find classical correlation between the qubit and the stochastic environment, which is quantified by the Shannon mutual information gained by performing an optimal measurement [4]. We also discuss the total correlation and quantum correlation [5]. Furthermore, we investigate the time evolution of the correlations in the cases of the fast and slow modulations of the dephasing. In Section 4, we provide concluding remarks.

2. The stochastic Liouville equation and its solution

We suppose that a quantum system is influenced by a classical environment which causes pure dephasing of the quantum system. The effect of the classical environment on the quantum system is described by means of a Markovian stochastic process, where the stochastic variable takes discrete values $\{a_1, a_2, \dots, a_n\}$. We denote as $\hat{W}(t, a_k)$ a compound state of the quantum system and the stochastic environment at time t [27,29], where $\hat{\rho}(t) = \sum_{k=1}^{n} \hat{W}(t, a_k)$ represents the reduced density matrix of the quantum system and $P(t, a_k) = \operatorname{Tr} \hat{W}(t, a_k)$ is the probability that the stochastic variable takes the value a_k at time t. Here Tr stands for the trace operation over the Hilbert space of the quantum system. Furthermore $\hat{\rho}(t|a_k) = \hat{W}(t, a_k)/P(t, a_k)$ is a conditional density matrix of the quantum system when the stochastic variable takes the value a_k [29]. If there is no correlation between the quantum system and the stochastic environment, the conditional density matrix $\hat{\rho}(t|a_k)$ is independent of the value a_k and thus we have $\hat{\rho}(t|a_k) = \hat{\rho}(t)$. The time evolution of the compound state $\hat{W}(t, a_k)$ is determined by the stochastic Liouville equation [27,29],

$$\frac{\partial}{\partial t}\hat{W}(t,a_k) = -\frac{i}{\hbar}\hat{H}^{\times}\hat{W}(t,a_k) - iV_k\hat{S}^{\times}\hat{W}(t,a_k) + \sum_{j=1}^{n}\Gamma_{kj}\hat{W}(t,a_j), \tag{1}$$

with $\hat{A}^{\times}\hat{B} = [\hat{A}, \hat{B}]$. In this equation, $\hat{H}$ is a Hamiltonian of the quantum system and $H_{\text{int}} = \hbar V_k \hat{S}$ represents the interaction Hamiltonian between the quantum system and stochastic environment with $\hat{S}$ being an appropriate operator of the quantum system. The parameter Γ_{kj} characterizes the Markov process by $\partial P(t, a_k)/\partial t = \sum_{j=1}^{n} \Gamma_{kj} P(t, a_j)$ [23], where the equality $\sum_{k=1}^{n} \Gamma_{kj} = 0$ is satisfied due to a conservation law of probability. In this Letter, we assume the two-state-jump Markov process [23]. Hence the stochastic variable takes two values $\pm\frac{1}{2}\Delta$. When $P(\infty, \frac{1}{2}\Delta) = P(\infty, -\frac{1}{2}\Delta) = \frac{1}{2}$, we can set $\Gamma_{11} = \Gamma_{22} = -\frac{1}{2}\gamma$, $\Gamma_{12} = \Gamma_{21} = -\frac{1}{2}\gamma$ [23,29]. We further assume that $V_1 = -V_2 = \frac{1}{2}\Delta$. Since we consider a pure dephasing process which conserves the energy of the quantum system, the operator $\hat{S}$ commutes with the Hamiltonian $\hat{H}$, that is, $[\hat{H}, \hat{S}] = 0$. Hence, in the rest of this Letter, we investigate the time evolution of the whole system in the interaction picture. For our purpose, it is convenient to introduce

$$\hat{W}(t) = \begin{pmatrix} \hat{W}_+(t) \\ \hat{W}_-(t) \end{pmatrix} = \begin{pmatrix} \hat{W}(t, \frac{1}{2}\Delta) \\ \hat{W}(t, -\frac{1}{2}\Delta) \end{pmatrix}. \tag{2}$$

Then we can write the stochastic Liouville equation as

$$\frac{\partial}{\partial t}\hat{W}(t) = \left[-i\Delta\hat{S}^{\times}\mathsf{K}_z - \frac{1}{2}\gamma + \frac{1}{2}\gamma(\mathsf{K}_+ + \mathsf{K}_-)\right]\hat{W}(t), \tag{3}$$

with

$$\mathsf{K}_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad \mathsf{K}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathsf{K}_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{4}$$

which are the generators of the SU(2) Lie algebra [32–34].

The stochastic Liouville equation given by Eq. (3) can be solved by means of the disentangling formula of the SU(2) Lie algebra [32–34] and the solution is

$$\begin{aligned}\hat{W}(t) &= \exp\left[-i\Delta t\hat{S}^{\times}\mathsf{K}_z - \frac{1}{2}\gamma t + \frac{1}{2}\gamma t(\mathsf{K}_+ + \mathsf{K}_-)\right]\hat{W}(0) \\ &= \begin{pmatrix} \hat{\mathcal{G}}_{++}(t) & \hat{\mathcal{G}}_{+-}(t) \\ \hat{\mathcal{G}}_{-+}(t) & \hat{\mathcal{G}}_{--}(t) \end{pmatrix}\begin{pmatrix} \hat{W}_+(0) \\ \hat{W}_-(0) \end{pmatrix},\end{aligned} \tag{5}$$

where the superoperators $\hat{\mathcal{G}}_{jk}(t)$ are given by

$$\hat{\mathcal{G}}_{++}(t) = e^{-\frac{1}{2}\gamma t}\left[\cosh\left(\frac{\gamma t}{2\hat{A}}\right) - i\left(\frac{\Delta\hat{S}^{\times}}{\gamma}\right)\hat{A}\sinh\left(\frac{\gamma t}{2\hat{A}}\right)\right], \tag{6}$$

$$\hat{\mathcal{G}}_{--}(t) = e^{-\frac{1}{2}\gamma t}\left[\cosh\left(\frac{\gamma t}{2\hat{A}}\right) + i\left(\frac{\Delta\hat{S}^{\times}}{\gamma}\right)\hat{A}\sinh\left(\frac{\gamma t}{2\hat{A}}\right)\right], \tag{7}$$

$$\hat{\mathcal{G}}_{+-}(t) = \hat{\mathcal{G}}_{-+}(t) = e^{-\frac{1}{2}\gamma t}\hat{A}\sinh\left(\frac{\gamma t}{2\hat{A}}\right), \tag{8}$$

with

$$\hat{A} = \left[1 - \left(\frac{\Delta\hat{S}^{\times}}{\gamma}\right)^2\right]^{-1/2}. \tag{9}$$

Then we obtain the reduced density matrix $\hat{\rho}(t) = \hat{W}_+(t) + \hat{W}_-(t)$ of the quantum system,

$$\begin{aligned}\hat{\rho}(t) = e^{-\frac{1}{2}\gamma t}&\left\{\cosh\left(\frac{\gamma t}{2\hat{A}}\right) + \left[1 - i\left(\frac{\Delta\hat{S}_z^{\times}}{\gamma}\right)\right]\hat{A}\sinh\left(\frac{\gamma t}{2\hat{A}}\right)\right\}\hat{W}_+(0) \\ &+ e^{-\frac{1}{2}\gamma t}\left\{\cosh\left(\frac{\gamma t}{2\hat{A}}\right) + \left[1 + i\left(\frac{\Delta\hat{S}_z^{\times}}{\gamma}\right)\right]\hat{A}\sinh\left(\frac{\gamma t}{2\hat{A}}\right)\right\}\hat{W}_-(0),\end{aligned} \tag{10}$$