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Physics Letters A 342 (2005) 263–266
PHYSICS LETTERS A
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Early nonlinear regime of MHD internal modes: The resistive case ☆

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Received 18 May 2005; received in revised form 20 May 2005; accepted 20 May 2005

Available online 31 May 2005

Communicated by F. Porcelli

Abstract

It is shown that the critical layer analysis, involved in the linear theory of internal modes, can be extended continuously into the early nonlinear regime. For the $m = 1$ resistive mode, the dynamical analysis involves two small parameters: the magnetic Reynolds number S and the $m = 1$ mode amplitude A, that measures the amount of nonlinearities in the system. The location of the instantaneous critical layer and the dominant dynamical equations inside it are evaluated self-consistently, as A increases and crosses some S-dependent thresholds. A special emphasis is put on the influence of the initial q-profile on the early nonlinear behavior. Predictions are given for a family of q-profiles, including the important low shear case, and shown to be consistent with recent experimental observations.

PACS: 52.30.Cv; 52.35.Py; 52.35.Mw; 52.55.Tn

Keywords: Magnetohydrodynamics; Nonlinear regime; Internal modes; Sawtooth oscillations

The $m = n = 1$ internal modes, such that the safety factor goes below one for some inner radius, remain critical macroscopic modes for large scale tokamak plasma dynamics and confinement. They are particularly involved in sawtooth oscillations and crashes. This is a common deleterious phenomenon as conventional tokamak discharges eventually operate with $q_0 < 1$ since current density tends to a peaked profile. Additionally, the $m = n = 1$ internal modes form a laboratory prototype for reconnection. Such phenomena typically proceed *beyond linear regime*.

We shall consider here the $m = n = 1$ purely resistive mode [1] that is ideally marginally stable. The original motivation of this work was to understand the growth of the $m = 1$ resistive mode up to its nonlinear saturation, on the basis of some striking numerical simulations performed by Aydemir [2] and

☆ Expanded version of a talk given at the 9th Plasma Easter Meeting, 30 March–1 April 2005, Villa Gualino, Turin.

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2005.05.052

previous observations [3]. Within the reduced MHD framework in cylindrical coordinates and some given q-profile [2], the time behavior of the kinetic energy in the $m = 1$ mode amounts to an initial exponential growth consistent with the linear regime, followed by a transient stage where the growth rate decreases, that is brutally interrupted by a second exponential growth in the nonlinear regime. This second exponential stage eventually terminates, as the kinetic energy in the $m = 1$ mode saturates which coincides with the completion of magnetic reconnection.

The reduced MHD system under consideration reads

$$\frac{\partial U}{\partial t} = [\phi, U] + [J, \psi], \tag{1}$$

$$\frac{\partial \psi}{\partial t} = [\phi, \psi] + \eta(J - J_0). \tag{2}$$

Helical symmetry is assumed: the poloidal and toroidal angles, respectively θ and φ, only come in through the helical angle $\alpha = \varphi - \theta$. ϕ and ψ are the plasma velocity and helical magnetic field potentials: the velocity is $\mathbf{v} = \hat{\boldsymbol{\varphi}} \times \nabla_\perp \phi$ and the magnetic field $\mathbf{B} = B_{0\varphi}\hat{\boldsymbol{\varphi}} + \hat{\boldsymbol{\varphi}} \times \nabla_\perp(\psi - r^2/2)$. $U = \nabla_\perp^2 \phi$ is the vorticity and $J = \nabla_\perp^2 \psi$ the helical current density, with $\nabla_\perp^2 \equiv r^{-1}\partial_r r \partial_r + r^{-2}\partial_\alpha^2$. Poisson brackets are defined by $[\phi, U] = -\hat{\boldsymbol{\varphi}} \cdot (\nabla_\perp \phi \times \nabla_\perp U) = r^{-1}(\partial_r \phi \partial_\alpha U - \partial_r U \partial_\alpha \phi)$. Eqs. (1), (2) are dimensionless: time has been normalized to the poloidal Alfvén time, the radial variable r to the minor radius, and η is the inverse of the magnetic Reynolds number S, and is given by the ratio of the poloidal Alfvén time to the resistive one. In high-temperature fusion plasmas, η is typically much smaller than one.

Consider equilibria such that, for some internal radius $r_{s0} < 1$, $q(r_{s0}) = 1$, that is $\psi_0'(r_{s0}) = 0$. Then, due to the Ohm's law (2), plasma volume divides in two region. Far from the $q = 1$ surface (outer domain), plasma behaves ideally whereas, in the vicinity of r_{s0} (inner region), resistivity plays a crucial, destabilizing, role. Linear theory [1] uses asymptotic matching analysis to provide $m = 1$ eigenfunctions in the form $A(t) f_L(r) \exp(i\alpha)$ valid in the whole domain. In the outer (ideal) domain, this solution is valid, that is nonlinear effects are negligible, as long as $A \ll 1$ [4]. Injecting the linear solutions $\psi_1(r, \alpha, t) = A(t)\psi_L(r)\exp(i\alpha)$ and $\phi_1(r, \alpha, t) = A(t)\phi_L(r)\exp(i\alpha)$ into (1), (2) calls for an amplitude expansion. The procedure has been given in Refs. [4,5]. The particularity of the linear radial eigenfunctions $\psi_L(r)$ and $\phi_L(r)$, that needs a careful consideration, is that they have strong gradients inside the critical layer. More precisely, their radial derivatives are of the order of the inverse of the critical layer width, that is $\mathcal{O}(\eta^{-1/3})$. This means in particular that this approach restricts to situations strictly above marginal stability and where the linear regime is well defined, with clear scalings, yielding the resistive ordering, and nonpathological q-profiles (in the sense of Ref. [6]).

We wish then to answer the question: "How does the $m = 1$ resistive mode develop into the nonlinear regime?" To do this, let us first recognize that the problem involves *two* small parameters. An obvious one is the resistivity η. However, considering it to be the only one small parameter, in some perturbation analysis with conventional expansions of the type $f = f_0 + \eta f_1 + \cdots$ would lead to a dead end: this would bring up a singular expansion, with additional $\eta \ln(\eta)$ terms, with no asymptotic validity unless assuming that the mode amplitude is always kept vanishingly small. It is interesting to note that such a procedure would actually be valid for the tearing mode with the small parameter limit Δ' [7,8]. In the present case, such a perturbation analysis would be ill-posed. A second small parameter enters the game, the $m = 1$ mode amplitude A that can be viewed as an indicator of the amount of nonlinearities in the system. As previously said, the approach will then be that of an amplitude expansion.

The first step will be to determine the end of validity of the linear regime. In the outer domain, this occurs for A of order one [4] but, in the inner domain, the linear solution breaks earlier. This occurs when mode coupling terms such as $[\phi_1, U_1]$ becomes of the same order as linear terms, that is for $A \gtrsim \eta^{2/3}$. At this point, $m = 0$ and $m = 2$ components begin to be "fed" nonlinearly by mode coupling terms: the $m = 0$ and $m = 2$ modes are nonlinearly driven. However, these mode coupling terms, quadratic in A, do not affect the $m = 1$ dynamics so that one could say that the $m = 1$ mode is still linear. At this stage, it is easy to check that the dominant equations on the $m = 1$ component are still the linear ones. This means that the radial structure of the solution should remain close to the linear one. Given that, it is possible to include the correc-