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Recursive proof of the Bell–Kochen–Specker theorem in any dimension $n > 3$

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Abstract

We present a method to obtain sets of vectors proving the Bell–Kochen–Specker theorem in dimension n from a similar set in dimension d ($3 \leqslant d < n \leqslant 2d$). As an application of the method we find the smallest proofs known in dimension five (29 vectors), six (31) and seven (34), and different sets matching the current record (36) in dimension eight.

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1. Introduction

The Bell–Kochen–Specker (BKS) theorem [1,2] states that quantum mechanics (QM) cannot be simulated by non-contextual hidden-variable theories. Any hidden-variable theory reproducing the predictions of QM must be *contextual* in the sense that the result of an experiment must depend on which other compatible experiments are performed jointly. The BKS theorem is independent of the state of the system, and is valid for systems described in QM by Hilbert spaces of dimension $d \geqslant 3$.

A proof of the BKS theorem consists of a set of physical yes–no tests, represented in QM by one-dimensional projectors, to which the rules of QM do not allow the assignment of predefined "yes" or "no" answers, regardless of how the system was prepared. In this Letter, yes–no tests will be represented by the vectors onto which the projectors project.

Several proofs of the BKS theorem in dimensions three, four and eight are known: see, for instance, [3] and the references in [4]. General procedures for extending the demonstration to a finite dimension n also

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exist [4–6]. In Section 2, we present a new method to obtain sets of vectors proving the BKS theorem in dimension n from a similar set in dimension d ($3 \leqslant d < n \leqslant 2d$). In Section 3 we compare this method with those of [4–6]. The main interest of this method is that it leads to the smallest proofs known in dimension five (29 vectors), six (31) and seven (34), and to different sets matching the current record (36) in dimension eight. These proofs are explicitly presented for the first time in Section 4; a preliminary version of them was referred to in [7,8].

Which one is the smallest number of yes–no tests needed to prove the BKS theorem in each dimension? This is an old question [4]. Recently, it has been proven that the answer is 18 for dimension four [9], and that there are no proofs with less yes–no tests in any dimension [10]. The proofs presented in Section 4 give an upper bound to this search in dimensions five to eight. The important point is that these bounds are sufficiently small so as to apply recently developed approaches capable to exhaustively explore all possible proofs of the BKS theorem [9,10]. The practical limitation of these approaches is that the complexity of the exploration grows exponentially with the number of vectors, making it difficult to explore all possible sets involving 30 vectors or more.

A set of n-dimensional vectors $X := \{\mathbf{u}_j\}_{j=1}^N$ is a proof of the BKS theorem if we cannot assign to each vector $\mathbf{u}_j$ a $v(\mathbf{u}_j)$ such that:

(a) Each $v(\mathbf{u}_j)$ has a uniquely defined value, 0 or 1 (“black” or “white”); this value is *non-contextual*, i.e., does not depend on which others $v(\mathbf{u}_k)$ are jointly considered.
(b) $\sum_{i=1}^n v(\mathbf{u}_i) = 1$ $\forall$ set of n mutually orthogonal vectors $\{\mathbf{u}_i\}_{i=1}^n \in X$.

In that case X is said to be “non-colourable”. A proof of the BKS theorem is said to be “critical” if all vectors involved are essential for the proof.

2. Recursive proof of the Bell–Kochen–Specker theorem

Let $A := \{\mathbf{a}_i\}_{i=1}^f$, $\mathbf{a}_i = (a_{i1}, \ldots, a_{id})$, be a proof in dimension d. For any $n := d + m$, $1 \leqslant m \leqslant d$, let us define two sets of n-dimensional vectors, $B^* := \{\mathbf{b}_i\}_{i=1}^f$, $C^* := \{\mathbf{c}_i\}_{i=1}^f$, obtained by appending to each vector $\mathbf{a}_i$ m zero components *on the right* and *on the left*, respectively; $\mathbf{b}_i := (a_{i1}, \ldots, a_{id}, 0, \ldots, 0)$, $\mathbf{c}_i := (0, \ldots, 0, a_{i1}, \ldots, a_{id})$. Let us also define the following sets of n-dimensional vectors: $\overline{B} := \{\mathbf{b}_j\}_{j=f+1}^{f+m}$, $b_{jk} := \delta_{j-f+d,k}$; $\overline{C} := \{\mathbf{c}_j\}_{j=f+1}^{f+m}$, $c_{jk} := \delta_{j-f,k}$; $B := B^* \cup \overline{B} = \{\mathbf{b}_j\}_{j=1}^{f+m}$, $C := C^* \cup \overline{C} = \{\mathbf{c}_j\}_{j=1}^{f+m}$.

Lemma. *B is BKS-colourable if and only if*

$$\sum_{j=f+1}^{f+m} v(\mathbf{b}_j) = 1. \tag{1}$$

Proof. The sets of d mutually orthogonal vectors in A become sets of n mutually orthogonal vectors in B, sharing the last m vectors, $\mathbf{b}_j \in \overline{B}$, $j = f+1, \ldots, f+m$. If condition (1) is fulfilled, we can colour B simply by assigning the values $v(\mathbf{b}_j) = 0$, $j = 1, \ldots, f$; conditions (a) and (b) are automatically satisfied. Conversely: if (1) is not verified, then $v(\mathbf{b}_j) = 0$, $j = f+1, \ldots, f+m$; the impossibility to colour set A in dimension d following rules (a), (b) implies the impossibility to colour B in dimension n. □

The same reasoning applies to C: C is colourable if and only if

$$\sum_{j=1}^{m} v(\mathbf{c}_j) = 1. \tag{2}$$

Theorem. *$D := B \cup C$ is a non-colourable set.*

Proof. If $d < n \leqslant 2d$, then $\overline{B} \cap \overline{C} = \emptyset$; conditions (1) and (2), necessary to colour B and C, would imply the existence of two mutually orthogonal vectors, $\mathbf{b}_k \in \overline{B}$, $\mathbf{c}_l \in \overline{C}$, with values $v(\mathbf{b}_k) = 1$, $v(\mathbf{c}_l) = 1$; this prevents $D = B \cup C$ from being coloured following rule (b); therefore D is a non-colourable set. □

The number g of different vectors in set D is $g \leqslant 2(f+m)$; the extreme is reached only if $B \cap C = \emptyset$. In general, set D is not critical (i.e., some subsets of D are also non-colourable sets). To search for critical subsets, we will use a generalization to arbitrary dimension of the computer program of Ref. [11].