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How similar is the performance of the cubic and the piecewise-linear circuits of Chua?

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ABSTRACT

This Letter reports phase diagrams quantifying and contrasting the dynamical performance of the paradigmatic piecewise-linear and cubic circuits of Chua. Although both circuits may be regarded as macroscopically isomorphic over wide regions in control parameter space, we show that their microscopic structure displays a myriad of rather distinctive intrinsic features making them unique. Inhomogeneities embedded in periodic and chaotic phases complicate some applications of the circuits but may also adequately act as realistic noise proxies in synchronization problems. In addition, infinite cascades of spirals and hubs observed experimentally very recently in a related dissipative flow are shown to be also present in both circuits of Chua, emerging however in a rather distinctive asymmetric way. Thus Chua's circuits may be used to study experimentally elusive and theoretically intricate phenomena generating periodicity hubs.

1. Introduction

The paradigmatic circuit of Chua has been continuously at the forefront of research during the last 25 years as a fruitful test-ground for theoretical and experimental advancements in non-linear dynamics which now fill several books [1–5] and many articles, e.g. [6–14]. Quite recently, this circuit has allowed the experimental observation of novel complex structures and phenomena in parameter space, like the so-called "shrimps" [15–18] which were observed both isolated [19,20] or, in a slightly different setup, forming infinite spirals connected to certain remarkable "periodicity hubs" [21,22,14,23]. The popularity of Chua's circuit is enhanced by the great reliability of electronic circuits and the excellent agreement normally found between measured and predicted behaviors. In fact, this characteristic of circuits allows one to probe novel devices and theories with very high accuracy [21–24].

According to a widespread opinion held about Chua's circuit (Fig. 1), both piecewise-linear and cubic circuits display dynamical behaviors which are "similar" [1–5]. Such similarities are usually elicited by comparing relevant dynamical quantities for a few selected parameters or, sometimes, by comparing bifurcation diagrams along specific sections of the control parameter space. However, some applications, e.g. synchronization of networks of identical and non-identical circuits, require a much more specific assessment of the degree of similarity of its constituents. Thus, synchronization of a large number of Chua's circuit leads one to ask the question whose answer is the focal point of this work: how quantitatively similar is the dynamical performance of circuits with piecewise-linear and cubic nonlinearities? In other words, how complete is the operational isomorphism of this pair of circuits? Could eventual differences in behavior act as realistic proxies for the ubiquitous noise seen in real-world systems? These are the question that we address here.

To quantify how similar both circuits behave we computed numerically high-resolution phase diagrams over extended parameter ranges (see Figs. 2–5 below). Synthetically, the general conclusion is that although on a coarse-grained scale both nonlinearities may be regarded as *macroscopically* isomorphic over wide regions in control space independently of the parameters tuned, their *microscopic* structure displays rather distinctive features.

For instance, while Chua's circuit contains periodicity hubs similar to the ones reported recently in lasers and other systems [14, 23–27], hubs in Chua's circuit present asymmetries and peculiarities which distinguishes them from everything seen so far (see below). A plethora of microscopic inhomogeneities between periodic and chaotic phases poses a number of challenges to efficient applications.

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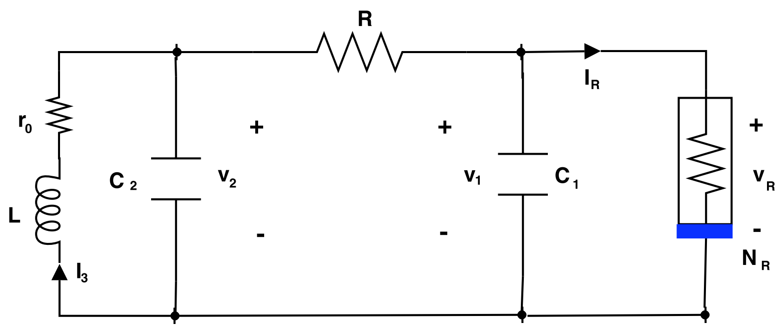

Fig. 1. The basic circuit leading to Eqs. (1a)–(1c), containing three control parameters $\alpha = C_2/C_1$, $\beta = R^2C_2/L$ and γ, defined as a function of the four linear elements C_1, C_2, L, and R in which the third parameter depends on the linear resistor r_0 in series with the inductor ($\gamma = Rr_0C_2/L$).

The aim of this work is to contrast the control parameter space of Chua's circuits with piecewise-linear and cubic nonlinearities. The main messages here are (i) that the widespread "equivalence" of both circuits is in fact not valid and their dynamical behavior needs to be carefully asserted for each specific application, and (ii) that the phase diagrams of both circuits display a rather rich structure, with many features which are not yet understood theoretically. Before starting, let us mention that there has been a number of recent investigations concerning the parameter space of both flavors of Chua's circuit, cubic and piecewise-linear [7–14], in particular concerning the identification of period-adding cascades in them [11,13].

2. The nonlinear circuits

As schematically shown in Fig. 1, Chua's circuit contains five linear elements (two capacitors, one inductor, and two resistors) and a nonlinear element, the so-called Chua's diode (N_R), playing the role of a negative resistor, and which normally contains two additional parameters [1–5]. In dimensionless form, the circuit is governed by the equations [6]:

$$\frac{dx}{dt} = \alpha\big(y - x - f(x)\big), \tag{1a}$$

$$\frac{dy}{dt} = x - y + z, \tag{1b}$$

$$\frac{dz}{dt} = -\beta y - \gamma z, \tag{1c}$$

where $f(x)$ stands for the nonlinearity and, in terms of the basic reactances, the three basic control parameters are

$$\alpha = \frac{C_2}{C_1}, \qquad \beta = \frac{R^2C_2}{L}, \qquad \gamma = \frac{Rr_0C_2}{L}. \tag{2}$$

Originally [28], the function $f(x)$ was taken as

$$f(x) = bx + \frac{1}{2}(a-b)\big(|x+1| - \big(|x-1|\big)\big), \tag{3}$$

where a and b are free parameters controlling the diode N_R. But a popular variant involves replacing this piecewise-linear function by diode with a smooth cubic characteristic [6]

$$f(x) = \hat{a}x^3 + \hat{b}x, \tag{4}$$

where $\hat{a}$ and $\hat{b}$ are free parameters. This cubic preserves the odd-symmetric character of the original piecewise function. The dynamical behavior of the piecewise-linear variant has been studied extensively and found not to capture correctly all features of a real circuit [29]. The main relevance of the cubic nonlinearity is that nonlinear devices are always smooth in real circuits [6]. Several works dealt with a smooth nonlinearity in Chua's oscillator [30–32]. The cubic nonlinearity has been implemented experimentally [29] and widely studied. For a comprehensive survey see Tsuneda [6].

To compare the performance of both circuits one first needs to ensure that they operate as identically as possible. To this end, we maximize the identity of the nonlinearities $f(x)$ above by suitably selecting the parameters $\hat{a}$ and $\hat{b}$ of the cubic to match a given pair (a, b). This is done by a least-square fit of the cubic to the piecewise linear function over an interval of approximation $[-d, d]$, a procedure which produces a parameter "bridge" among both circuits [6,33,34]:

$$\hat{a} = -35\big(d^2-1\big)^2(a-b)/\big(16d^7\big), \tag{5a}$$

$$\hat{b} = b + \big(45d^4 - 50d^2 + 21\big)(a-b)/\big(16d^5\big). \tag{5b}$$

Following previous workers [2], we fix $a = -8/7 \simeq -1.1428$, $b = -5/7 \simeq -0.7142$, and $d = 2$ a choice that gives $\hat{a} = 0.0659$ and $\hat{b} = -1.1671$.

3. Phase diagrams

We start now by illustrating how changes in the reactances affect solutions and stability of both circuits by computing and comparing Lyapunov phase diagrams [14] for them. Of particular interest is to compare changes in shape and volume of both periodic and chaotic phases and, more importantly, the details of their inner structure.

Fig. 2 shows phase diagrams illustrating relevant portions of the $\alpha \times \gamma$ parameter plane for each circuit, a plane containing particularly rich mixture of periodicity and chaotic phases. As indicated by the color scales, gray shadings signal parameter regions characterized by periodic solutions (negative Lyapunov exponents), while colors always mark chaotic phases (positive exponents). The bluish coloration in Fig. 2(a) represents the piecewise linear circuit while the greenish hue is used for the cubic nonlinearity. Each panel in our figures displays $1200 \times 1200 = 1.44 \times 10^6$ Lyapunov exponents. The large pink domains indicate parameters for which most initial conditions lead to unbounded solutions. Fig. 2 was computed for $\beta = 1000$ and presents "asymptotic" phase diagrams, in the sense that they essentially remain invariant when β is further increased. The basic structure in both panels of Fig. 2 show that while the overall coarse-grained distribution of chaos and periodicity in Fig. 2 looks similar at first sight, their precise distribution has a large number of small differences that are hard to summarize efficiently with words.

Fig. 3 illustrates parameter isomorphism when tuning β from 10 to 100, i.e. when moving from the low β region to the asymptotic limits shown in Fig. 2. Again, while there is an overall agreement of the dynamics observed for both circuits when increasing β, the phase diagrams are not identical. In lose words, while it