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Short-time correlations of many-body systems described by nonlinear Fokker–Planck equations and Vlasov–Fokker–Planck equations

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Abstract

Analytical expressions for short-time correlation functions, diffusion coefficients, mean square displacement, and second order statistics of many-body systems are derived using a mean field approach in terms of nonlinear Fokker–Planck equations and Vlasov–Fokker–Planck equations. The results are illustrated for the Desai–Zwanzig model, the nonlinear diffusion equation related to the Tsallis statistics, and a Vlasov–Fokker–Planck equation describing bunch particles in particle accelerator storage rings.

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1. Introduction

Mean field theory is a fundamental approach to study stochastic properties of many-body systems. In particular, using mean field theory first order statistics of many-body systems can be determined in terms of stationary and transient distribution functions. Famous examples of dynamical mean field models are the Boltzmann equation [1], Vlasov equations [2–6], and nonlinear Fokker–Planck equations [7–26]. These equations have in common that they are nonlinear with respect to density measures. The nonlinearities reflect the interactions between the subsystems of the corresponding many-body systems.

While the first order statistics of dynamical mean field models has extensively been studied, comparatively little is known about the second order statistics and time correlation functions. In this context, Vlasov–Fokker–

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Planck equations [27–36], and nonlinear Fokker–Planck equations are of particular interest because they allow for an interpretation in terms of generalized self-consistent Langevin equations [23,37–41]. That is, trajectories of Brownian particles can be computed for this kind of dynamical mean field models. Consequently, as far as Vlasov–Fokker–Planck equations and nonlinear Fokker–Planck equations are concerned, second order statistics and time correlation functions can at least be determined numerically by solving appropriately defined Brownian dynamics models.

Analytical expressions for time correlation functions have been derived previously in the context of generalized fluctuation–dissipation theorems [42–44]. In addition, a hierarchy of differential equations has been derived involving time correlation functions of different order. In some cases this hierarchy is closed and can be solved, in others it cannot [45]. Finally, there are some special cases in which exact expressions for two time-point joint probability densities of dynamical mean field models can be derived, that is, in which analytical expressions for the second order statistics of dynamical mean field models can be found [14,46]. However, a general analytical discussion of the second order statistics of mean field models in terms of nonlinear Fokker–Planck equations and Vlasov–Fokker–Planck equations has not been carried out so far.

In the present study, we will derive analytical expressions for second order statistics and time correlation functions for short time differences. In doing so, we will also determine diffusion coefficients relevant on short time scales. The focus will be on dynamical mean field models given in terms of nonlinear Fokker–Planck equations (Sections 2.1 and 2.2) and Vlasov–Fokker–Planck equations (Sections 2.3 and 2.4).

2. Short-time correlation functions and second order statistics

2.1. Nonlinear Fokker–Planck equations: general case

In line with mean field theory, we describe many-body systems in terms of the behavior of a single subsystem (μ-space description). Accordingly, let $X(t) \in \Omega$ denote a time-dependent state variable of a subsystem of a many-body systems, where $X(t)$ is defined on a single subsystem phase space Ω at every time point t. We assume that $X(t)$ has distribution $u(x)$ at an initial time $t = t_0$ and denote that single subsystem probability density by $P(x,t;u) = \langle \delta(x - X(t)) \rangle$, where $\langle \cdot \rangle$ is an ensemble average and $\delta(\cdot)$ is the delta function. We further assume that $P(x,t;u)$ satisfies a nonlinear Fokker–Planck equation of the form

$$\frac{\partial}{\partial t} P(x,t;u) = -\frac{\partial}{\partial x} D_1(x,P)P + \frac{\partial^2}{\partial x^2} D_2(x,P)P. \tag{1}$$

In what follows, the focus will be on many-body systems that can be described in terms of strongly nonlinear Fokker–Planck equations [14,41,47], which means that the transition probability density $P(x,t|x',t';u)$ of $X(t)$ satisfies

$$\frac{\partial}{\partial t} P(x,t|x',t';u) = -\frac{\partial}{\partial x} D_1\big(x, P(x,t)\big) P(x,t|x',t';u) + \frac{\partial^2}{\partial x^2} D_2\big(x, P(x,t)\big) P(x,t|x',t'). \tag{2}$$

In particular, in the stationary case, we have $P(x,t;u) = P_{\mathrm{st}}(x)$ and

$$\frac{\partial}{\partial t} P_{\mathrm{st}}(x,t|x',t';u = P_{\mathrm{st}}) = \left[-\frac{\partial}{\partial x} D_1\big(x, P_{\mathrm{st}}(x)\big) + \frac{\partial^2}{\partial x^2} D_2\big(x, P_{\mathrm{st}}(x)\big)\right] P_{\mathrm{st}}(x,t|x',t';P_{\mathrm{st}}). \tag{3}$$

2.1.1. Natural boundary conditions

Assuming natural boundary condition and $\Omega = \mathbb{R}$, the transition probability density $P_{\mathrm{st}}(x,t'+\Delta t|x',t';P_{\mathrm{st}})$ for small time differences Δt is determined by the short-time propagator $P^{(0)}(x,\Delta t|x')$ like $P_{\mathrm{st}}(x,t'+\Delta t|x',t';P_{\mathrm{st}}) =$