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The asymptotic iteration method for the eigenenergies of the anharmonic oscillator potential $V(x) = Ax^{2\alpha} + Bx^2$

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Abstract

The asymptotic iteration method is used to calculate the eigenenergies for the anharmonic oscillator potentials $V(x) = Ax^{2\alpha} + Bx^2$ ($A > 0$, $B < 0$), with ($\alpha = 2$) for quartic, and ($\alpha = 3$) for sextic anharmonic oscillators. An adjustable parameter β is introduced in the method to improve its rate of convergence. Comparing the present results with the numerical values calculated by earlier workers, it is found that, asymptotically this method gives exact results over the full range of parameter values, A and B.

1. Introduction

It is well known that the exact solution of the Schrödinger wave equation has been possible only for a handful of potentials and in most cases one has to resort to numerical techniques or approximation schemes. In recent years, the study of anharmonic oscillator potentials with higher order anharmonicities has become much more desirable for the theoretical understanding of several newly discovered phenomena in different branches of physics [1–3]. However, the lack of an exact analytical solution to the anharmonic oscillator potentials, make these potentials one of the most popular theoretical laboratories for examining the validity of various approximation techniques based on perturbative, and nonperturbative approaches [4–8]. Two well-known and important examples, in this respect, are the quartic, and sextic anharmonic oscillator potentials $V(x) = Ax^{2\alpha} + Bx^2$ ($A > 0$, $B < 0$) ($\alpha = 2, 3$) which provoke special interest [9,10]. These potentials play an important role in the quantum study of tunneling time problems, in spectra of molecules such as ammonia and hydrogen-bonded solids, and in field theory. Moreover, they might be used as a potential model for quark confinement in quantum chromodynamics [11].

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As is well known, the usual Rayleigh–Schrödinger perturbation method gives a divergent series for these potentials, whatever the intensity of the anharmonic interaction [12]. Therefore, some other methods using Pade approximations [13], Borel summation [14], Symanzik scaling symmetry [10], and quasi-exact solutions [9] were introduced to find the eigenenergies (E_n) of the quartic, and sextic anharmonic oscillator potentials. However, the obtained number of analytical states were limited, and the potential parameters in those cases had to obey certain relations [9].

Thus, the need arises to have a relatively simple and effective method which will give, to a high degree of accuracy, the eigenenergies, and eigenfunctions without any constraint on the potential parameter values involved. Therefore, this work applies a new and efficient method: the asymptotic iteration method (AIM) [15,16] for the computation of the eigenenergies of both the quartic, and sextic anharmonic oscillator potentials. This method is very easy to implement and the results are sufficiently accurate for practical purposes. In a single batch one is able to study not only the eigenenergies, but also the eigenfunctions.

Furthermore, the asymptotic iteration method puts no constraint on the potential parameter values involved, like A and B. It also handles E_n with large n which used to pose many numerical instabilities to some of the previously mentioned methods. Therefore, the main motivation of the present work is to overcome the shortcomings of those approaches, and to formulate an elegant algebraic approach that yields a very simple analytic formula which will rapidly give the eigenenergies to a high degree of accuracy. To achieve this goal we explore the use of an adjustable parameter β introduced in [17], for improving the rate convergence of the AIM.

With this in mind, this Letter is organized as follows. In Section 2, the asymptotic iteration method for the eigenenergies of the anharmonic oscillator potentials is outlined. The analytical expressions for asymptotic iteration method are cast in such a way that allows the reader to use them without proceeding into their derivation. In Section 3, we explain the method to obtain numerically the eigenenergies, and present our numerical results compared with other existing solutions. Finally, we remark on the results and our findings.

2. The asymptotic iteration method for the anharmonic oscillator potentials

In order to calculate the eigenenergies and eigenfunctions of the anharmonic oscillator potentials, one needs to solve the eigenvalue equation (in units $\hbar = m = 1$)

$$\left[-\frac{1}{2}\frac{d^2}{dx^2} + V(x)\right]\Psi_n(x) = E_n\Psi_n(x), \tag{1}$$

in which E_n are the eigenenergies, and $V(x)$ is the anharmonic oscillator potential

$$V(x) = Ax^{2\alpha} + Bx^2 \quad (A > 0,\ B < 0), \quad \text{with } (\alpha = 2, 3). \tag{2}$$

We recall that Eq. (1) is the one-dimensional Schrödinger equation if x is defined on the whole line ($-\infty < x < +\infty$) with eigenfunctions normalized as $\int_{-\infty}^{+\infty} |\Psi_n(x)|^2\,dx = 1$. In order to guarantee the asymptotic behaviour of the Schrödinger equation when $x \to \infty$, so that

$$\frac{d^2\Psi_n(x)}{dx^2} \to 2\left(Ax^{2\alpha} + Bx^2\right)\Psi_n(x) \quad (x \to \infty), \tag{3}$$

then $\Psi_n(x)$ should look like $\Psi_n(x) = e^{-\gamma x^{\alpha+1}}$.

Substituting the above eigenfunctions in the corresponding asymptotic equation, one gets the following relation among the arbitrary parameter introduced in the eigenfunctions γ and the potential parametre A: $\gamma = \frac{\sqrt{2A}}{\alpha+1}$.

This asymptotic behaviour suggests a suitable ansatz for the eigenfunctions $\Psi_n(x)$,

$$\Psi_n(x) = e^{-\frac{\sqrt{2A}}{\alpha+1}x^{\alpha+1}} f_n(x). \tag{4}$$