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A superconvergent universality induced by non-associativity

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ABSTRACT

The star products in symbolic dynamics, as effective algebraic operations for describing self-similar bifurcation structure in classical dynamical systems, are found to have either associativity or non-associativity. In this Letter, non-associative star products in trimodal iterative dynamical systems are considered. As the left and right operations have different effects, right-associative star products break the conventional Feigenbaum's metric universality. Through high precision parallel computation, it is found that period-p-tupling bifurcation processes described by right-associative star products exhibit a superconvergent universality of double exponential form.

1. Introduction

The discovery of Feigenbaum's universal constants α (scaling factor) and δ (convergence rate) [1,2] is a milestone in the development of nonlinear science. Independent of specific maps, these two constants certainly represent a physical universality. The former describes the self-similarity of orbits of strange attractors in the phase space; the latter describes the successive ratio of parameters of period-doubling or period-p-tupling bifurcations. Although discovered directly in iterative dynamical systems, they also exist in physical systems (such as the Rayleigh–Bénard system [3], the forced Brusselator [4] and the periodically driven Rössler oscillator [5]) when these systems are transformed by the Poincaré sections into lower-dimensional systems, or iterative systems become higher-dimensional physical systems by suspensions. Moreover, they were proved to be nature's numbers in mathematics [6], and observed in experiments of superfluid helium-4 [7], dripping faucet [8], pendulum [9], electric circuit [10], and dynamics of a railway wheelset [11], etc. More examples can be found in Ref. [12].

Since being discovered in the late 1970s, Feigenbaum's constants have greatly stimulated theoretical studies of nonlinear physical systems. Feigenbaum introduced Wilson's renormalization group in critical phenomena into the self-similar renormalization group equation in the critical area of phase orbits of nonlinear systems, which determines a fixed point in the functional space [13]. This functional renormalization group equation has important theoretical significance: the universality for describing self-similarity is contained in this equation which relates the two universal constants via itself and its linearized equation. As different universal constants ($\alpha(W)$ and $\delta(W)$) are corresponding to different symbolic sequences W (with different renormalization group equations), they provide a new degree of freedom [14] for the study of nonlinear systems. Theoretical extensions along this direction are the cycle expansion [15] and the Riemann zeta-function theory [16] of chaotic systems. Thus Feigenbaum's constants can be theoretically analyzed and calculated.

Before developing such renormalization group analyses, an effective approach to calculating universal constants is to find more star products in symbolic dynamics which can describe self-similar orbits and are the inverse procedure of renormalizations. To study universality of period-p-tupling bifurcations in multimodal maps, the Derrida–Gervois–Pomeau (DGP) star product in unimodal maps [17] has been generalized to dual star products in bimodal maps [18,19] and cyclic star products in trimodal and quadrumodal maps [20,21], respectively. As important algebraic tools for inverse renormalizations in symbolic space, the generalized normal star products are associative and preserve the topological entropy [22–24, 19,25,26]. These two features ensure Feigenbaum's constants (convergence rates or bifurcation "speeds") to be first-order geometric ratios.

However, with the rapid progress of high speed and high precision computation as well as the generalization of star products,

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abnormal phenomena appear: the associative law of star products is broken and Feigenbaum's successive ratios (bifurcation "speeds") turn out to be divergent [27]. In this Letter, our numerical results show that these superconvergent bifurcation processes are double exponential, which can be described by right-associative non-normal star products and regarded as the second-order effect of Feigenbaum's universality; whereas the conventional Feigenbaum's constants, appearing as geometric ratios under the control of equal topological entropy classes, represent the first-order (i.e. exponential) effect of universality. It is known that non-associative algebras, such as the Lie algebra and the Jordan algebra, have been widely used in physics, especially in quantum mechanics, but they are rare in classical dynamical systems. Our finding requires a stronger associative order of physical operators, just like noncommutativity should be considered in the study of some fine effects in quantum mechanics, such as the Lamb shift and the Zeeman effect of hydrogen [28]. Although physical systems are commonly associative, this Letter definitely reveals a phenomenon of non-associative physical quantities, and indicates that its superconvergent universality has a double exponential form. This superconvergent effect does not appear in unimodal and bimodal systems; it can only be observed when systems become trimodal and even m-modal ($m > 3$) and are studied by high precision numerical experiment. It would be possible, for example, to detect such superconvergent effect in some fine physical systems (such as circuit systems [29]) if equipped with suitable bifurcation controls [30]. The appearance of non-associativity in nonlinear dynamics is an interesting phenomenon worth being investigated, which may lead to new research beyond Feigenbaum's universality.

This Letter is organized as follows: In Section 2, we briefly recall symbolic dynamics of trimodal maps, and then present the divergence phenomenon of Feigenbaum's universality with an example of period-tripling bifurcation. In Section 3, we analyze this divergence and give the superconvergent universality of a double exponential form. Finally in Section 4, we discuss the possible reasons of occurrence of this superconvergence by analyzing the differences between non-associative star products and associative ones from the viewpoint of the difference in structures of the orbits and in behaviors of the topological entropy invariants.

2. Divergence of generalized Feigenbaum's successive ratios

We first briefly recall some basic notions of symbolic dynamics of trimodal maps. Consider an arbitrary trimodal map $f_{\mu,\nu,\xi}: I \to I$ on the real interval $I = [-1, 1]$, which depends on three parameters $(\mu, \nu, \xi) \subset \mathbb{R}^3$ and can be written as $x_{j+1} = f_{\mu,\nu,\xi}(x_j)$. The map f (simplified notation for $f_{\mu,\nu,\xi}$) has three critical (or turning) points denoted as $C_1 = C, C_2 = D, C_3 = E$ and four monotone branches as L, M, N, R, respectively (supposing L to be a monotone increasing branch, i.e., the shape of f to be $(+-+-)$ type). Let c_1, c_2, c_3 be coordinates of three critical points, then we have $-1 < c_1 < c_2 < c_3 < 1$. Obviously, the symbolic order $L < C < M < D < N < E < R$ is a natural order.

For a trimodal map f, starting from an initial point x_0, a numerical orbit $(x_0, x_1, \dots, x_j, \dots)$ can be obtained by iterating f, while a symbolic sequence $W = w_0 w_1 \dots w_j \dots$ is assigned by the following coarse-grained description: $w_j = L, C, M, D, N, E, R$ for $x_j \in [-1, c_1)$, $x_j = c_1$, $x_j \in (c_1, c_2)$, $x_j = c_2$, $x_j \in (c_2, c_3)$, $x_j = c_3$, $x_j \in (c_3, 1]$, respectively. A sequence starting from a critical point is called a *kneading* sequence. A periodic sequence passing through all the three critical points is called a *triply superstable kneading* (TSSK) sequence. In this Letter, the symbolic space $\Sigma_4^{\mathbb{Z}^+}$ refers to the set of all TSSK sequences of alphabet $\Sigma_4 = \{L, C, M, D, N, E, R\}$.

According to the ordering of appearances of critical points, TSSK sequences can be divided into six types: $ZEXDYC$, $YCZEXD$, $XDYCZE$, $XDZEYC$, $ZEYCXD$ and $YCXDZE$, where X, Y and Z are sequences of symbols L, M, N and R. Correspondingly, there are also six types of cyclic star products: $(E|D|C)$-, $(C|E|D)$-, $(D|C|E)$-, $(D|E|C)$-, $(E|C|D)$- and $(C|D|E)$-type (cf. [20] for the specific composition rules or multiplication tables). For simplicity, here we only present the composition rules of the $(E|D|C)$-type star product. For any two given TSSK sequences $W_1 = Z_1 E X_1 D Y_1 C$ and $W_2 = Z_2 E X_2 D Y_2 C = w_1^2 w_2^2 \dots w_j^2 \dots w_{|W_2|}^2$, where $w_j^2 \in \Sigma_4$ and $|W_2|$ denotes the period or length of W_2, the star product $W_1 * W_2$ can be calculated as

$$\begin{aligned} W_1 * W_2 &= Z_1 E X_1 D Y_1 C * Z_2 E X_2 D Y_2 C \\ &= \bigcup_{j=1}^{|W_2|} \left(Z_1 E X_1 D Y_1 C * w_j^2\right) \\ &= \bigcup_{j=1}^{|W_2|} Z_1 \left(E * w_j^2\right)^{\tau(Z_1)} X_1 \left(D * w_j^2\right)^{\tau(X_1)} Y_1 \left(C * w_j^2\right)^{\tau(Y_1)}, \end{aligned}$$

where $\cup$ means the concatenation of subsequences ($U \cup V = UV$); $\tau(Z_1)$, $\tau(X_1)$ and $\tau(Y_1)$ are parities of subsequences Z_1, X_1 and Y_1 in $W_1 = Z_1 E X_1 D Y_1 C$ defined as: $\tau(W) = +$ if $J(W)$ is even, and $\tau(W) = -$ if $J(W)$ is odd, where $J(W)$ is the number of appearances of letter M and R in a sequence W. To obtain a $(E|D|C)$-type star product, the procedure of multiplication of letters needs to be carried out according to the following composition rules: $E * L = E * C = E * M = E * D = E * N = E^-$, $E * E = E^0$, $E * R = E^+$; $D * L = D * C = D * M = D^-$, $D * D = D^0$, $D * N = D * E = D^+$, $D * R = D^-$; $C * L = C^+$, $C * C = C^0$, $C * M = C * D = C * N = C^-$, $C * E = C * R = C^+$; finally, C^-, C^0, C^+, D^-, D^0, D^+, E^-, E^0 and E^+ are translated into letters L, C, M, D, N, E and R, respectively: $C^- = L$, $C^0 = C$, $C^+ = D^- = M$, $D^0 = D$, $D^+ = E^- = N$, $E^0 = E$, and $E^+ = R$. For example, we can easily get $EDC * EDC = ENMNDLNMC$.

Let W_1, W_2 and W_3 be TSSK sequences, we call a star product $*$ an *associative* star product if the associativity $(W_1 * W_2) * W_3 = W_1 * (W_2 * W_3)$ holds; otherwise a *non-associative* star product. Computer experiments show that only $(C|E|D)$- and $(E|C|D)$-type star products are associative, all the other four types of cyclic star products are non-associative. For instance, for the $(E|D|C)$-type star product mentioned above, one can easily get

$$\begin{aligned} &(EDC * EDC) * EDC \\ &\quad = ENMNMLNMLNNMNDLNMMNNMNNLNMC, \end{aligned}$$

and

$$\begin{aligned} &EDC * (EDC * EDC) \\ &\quad = ENMNNLNMLNNLNDLNMMNNLNMLNMC. \end{aligned}$$

Obviously $(EDC * EDC) * EDC \neq EDC * (EDC * EDC)$. The non-associative star products are first found in trimodal maps [27]; in fact, they always exist in m-modal maps for $m \geqslant 3$.

It is known that a period-p-tupling bifurcation process can be described by self star products W^{*n} of a TSSK sequence W (with $p = |W|$). Once the associativity of algebraic products is broken, we need to consider the left- and right-multiplications of algebraic products, respectively. Here, we denote left-associativity (of non-associative star products) as

$$W_l^{*n} = W_l^{*(n-1)} * W, \tag{1}$$

and right-associativity as

$$W_r^{*n} = W * W_r^{*(n-1)}. \tag{2}$$