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Spin glasses and nonlinear constraints in portfolio optimization

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ABSTRACT

We discuss the portfolio optimization problem with the obligatory deposits constraint. Recently it has been shown that as a consequence of this nonlinear constraint, the solution consists of an exponentially large number of optimal portfolios, completely different from each other, and extremely sensitive to any changes in the input parameters of the problem, making the concept of rational decision making questionable. Here we reformulate the problem using a quadratic obligatory deposits constraint, and we show that from the physics point of view, finding an optimal portfolio amounts to calculating the mean-field magnetizations of a random Ising model with the constraint of a constant magnetization norm. We show that the model reduces to an eigenproblem, with $2N$ solutions, where N is the number of assets defining the portfolio. Also, in order to illustrate our results, we present a detailed numerical example of a portfolio of several risky common stocks traded on the Nasdaq Market.

1. Introduction

Portfolio optimization is an important problem in economic analysis and risk management [1,2], and under certain nonlinear constraints maps exactly into the problem of finding the ground states of a long-range spin glass [3–5]. The main assumption is that the return of any financial asset is described by a random variable, whose expected mean and variance are interpreted as the reward, and respectively the risk of the investment. The problem can be formulated as follows: given a set of financial assets, characterized by their expected mean and their covariances, find the optimal weight of each asset, such that the overall portfolio provides the smallest risk for a given overall return. The standard mean-variance optimization problem has a unique solution describing the so called "efficient frontier" in the (*risk*, *return*)-plane [6]. The expected return is a monotonically increasing function of the standard deviation (risk), and for accepting a larger risk the investor is rewarded with a higher expected return. Recently it has been shown that the portfolio optimization problem containing short sales with obligatory deposits (margin accounts) is equivalent to the problem of finding the ground states of a long-range Ising spin glass, where the coupling constants are related to the covariance matrix of the assets defining the portfolio [3–5]. As a consequence of this nonlinear constraint, the solution consists of an exponentially large number of optimal portfolios, completely different from each other, and extremely sensitive to any changes in the input parameters of the problem. Therefore, under such constraints, the concept of rational decision making becomes questionable, since the investor has an exponential number of "options" to choose from. Here, we discuss the portfolio optimization problem using a quadratic formulation of the nonlinear obligatory deposits constraint. From the physics point of view, finding an optimal portfolio amounts to calculating the mean-field magnetizations of a random Ising model with the constraint of a constant magnetization norm. We show that the proposed model reduces to an eigenproblem, with $2N$ solutions, where N is the number of assets defining the portfolio. In support to our results, we also work out a detailed numerical example of a portfolio of several risky common stocks traded on the Nasdaq Market.

2. Nonlinear optimization model

A portfolio is an investment made in N assets A_n, with the expected returns r_n, and covariances $s_{nm} = s_{mn}$, $n, m = 1, 2, \dots, N$. Let w_n denote the relative amount invested in the n-th asset. Negative values of w_n can be interpreted as short selling. The variance of the portfolio captures the risk of the investment, and it is given by:

$$s^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j s_{ij} = \mathbf{w}^T \mathbf{S}\mathbf{w}, \tag{1}$$

where $\mathbf{w} = [w_1, w_2, \dots, w_N]^T$ is the vector of weights, and $\mathbf{S} = [s_{nm}]$ is the covariance matrix. Also, another characteristic of the portfolio is the expected return:

$$\rho = \sum_{n=1}^{N} w_n r_n = \mathbf{w}^T \mathbf{r}, \tag{2}$$

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where $\mathbf{r} = [r_1, r_2, \ldots, r_N]^T$ is the vector of asset returns. The standard portfolio selection problem consists in finding the solution of the following multi-objective optimization problem [1,2,6]:

$$\min_{\mathbf{w}} \{ s^2 = \mathbf{w}^T \mathbf{S} \mathbf{w} \}, \tag{3}$$

$$\max_{\mathbf{w}} \{ \rho = \mathbf{w}^T \mathbf{r} \}, \tag{4}$$

subject to the invested wealth constraint:

$$\sum_{n=1}^{N} w_n = 1. \tag{5}$$

As mentioned in the introduction, this problem has a unique solution, which can be obtained using the method of Lagrange multipliers [1,2,6].

Recently it has been shown that by replacing the invested wealth constraint (5) with an obligatory deposits constraint the problem cannot be solved analytically anymore [3–5]. The constraint consists in imposing the requirement to leave a certain deposit (margin) proportional to the value of the underlying asset, and it has the form:

$$\gamma \sum_{n=1}^{N} |w_n| = W, \tag{6}$$

where $\gamma > 0$ is the fraction defining the margin requirement, and W is the total wealth invested. As a direct consequence of the constraint's nonlinearity, the problem has an exponentially large number of solutions:

$$n(N, \rho) \sim \exp\big(\omega(\rho) N\big), \tag{7}$$

where $\omega(\rho)$ is a positive number depending on the portfolio return [3–5]. The solutions are also completely different from each other, and extremely sensitive to any changes in the input parameters of the problem. Thus, finding the global optimum becomes prohibitive (NP-problem) for a larger N.

Let us now reformulate this constraint using a quadratic function:

$$\gamma \sum_{n=1}^{N} w_n^2 = W. \tag{8}$$

Thus, we impose the requirement to leave a certain deposit proportional to the quadratic value of the asset. This is equivalent to a constant norm $\|\mathbf{w}\|^2 = k = W/\gamma$. Also, we combine the multi-objective optimization problem into a single Lagrangian objective function as follows:

$$\min_{\mathbf{w},\mu} \big\{ F(\mathbf{w}, \lambda, \mu) = \lambda \mathbf{w}^T \mathbf{S} \mathbf{w} - (1-\lambda) \mathbf{w}^T \mathbf{r} - \mu \big(\mathbf{w}^T \mathbf{w} - k\big) \big\}, \tag{9}$$

where $\lambda \in [0, 1]$ is the risk aversion parameter, and μ is the Lagrange parameter.

If $\lambda = 0$ then the solution corresponds to the portfolio with maximum return, without considering the risk. In this case the optimal solution will be formed only by the asset with the greatest expected return. The case with $\lambda = 1$ corresponds to the portfolio with minimum risk, regardless of the value of the expected return. In this case the problem becomes:

$$\min_{\mathbf{w},\mu} \big\{ F(\mathbf{w}, 1, \mu) = \mathbf{w}^T \mathbf{S} \mathbf{w} - \mu \big(\mathbf{w}^T \mathbf{w} - k\big) \big\}, \tag{10}$$

with the solutions given by the equation:

$$\nabla_{\mathbf{w}} F(\mathbf{w}, \lambda, \mu) = 2\mathbf{S}\mathbf{w} - 2\mu \mathbf{w} = 0. \tag{11}$$

This is a standard eigenproblem:

$$\mathbf{S}\mathbf{w} = \mu \mathbf{w}, \tag{12}$$

where $\mathbf{S}$ is a symmetric matrix with N real eigenvalues, and N real eigenvectors. The eigenvector corresponding to the largest eigenvalue will provide the global optimum, since it will have the lowest risk.

Any value $\lambda \in (0, 1)$ represents a tradeoff between the risk and return. In this case the solution corresponds to the critical point of the Lagrangian, which is also the solution of the following system of equations:

$$\nabla_{\mathbf{w}} F(\mathbf{w}, \lambda, \mu) = 2\lambda \mathbf{S}\mathbf{w} - (1-\lambda)\mathbf{r} - 2\mu \mathbf{w} = 0, \tag{13}$$

$$\frac{\partial F(\mathbf{w}, \lambda, \mu)}{\partial \mu} = \mathbf{w}^T \mathbf{w} - k = 0. \tag{14}$$

One can see that the Lagrangian objective function is equivalent to the free energy of an Ising model with random couplings $J_{nm} = -2\lambda s_{nm}$ and a random magnetic field $h_n = (1-\lambda) r_n$. From the physics point of view, finding an optimal portfolio amounts to calculating the mean-field magnetizations w_n of this random Ising model with the constraint of a constant magnetization norm. In the following we show that solving this system of equations reduces to an inhomogeneous eigenproblem.

From the first equation we have:

$$\mathbf{w} = \frac{1}{2}(1-\lambda)(\lambda \mathbf{S} - \mu \mathbf{I})^{-1} \mathbf{r}. \tag{15}$$

Introducing this result into the second equation we obtain:

$$\frac{1}{4}(1-\lambda)^2 \mathbf{r}^T (\lambda \mathbf{S} - \mu \mathbf{I})^{-2} \mathbf{r} - 1 = 0. \tag{16}$$

The left-hand side of this equation is the Schur complement of the matrix:

$$\mathbf{M} = \begin{bmatrix} (\lambda \mathbf{S} - \mu \mathbf{I})^2 & \frac{1}{2}(1-\lambda)\mathbf{r} \\ \frac{1}{2}(1-\lambda)\mathbf{r}^T & 1 \end{bmatrix}. \tag{17}$$

Since this matrix must be singular (the Schur complement is zero), we have:

$$\det\left[(\lambda \mathbf{S} - \mu \mathbf{I})^2 - \frac{1}{4}(1-\lambda)^2 \mathbf{r}\mathbf{r}^T \right] = 0, \tag{18}$$

which reduces to:

$$\det\left[\frac{1}{4}(1-\lambda)^2 \mathbf{r}\mathbf{r}^T - \lambda^2 \mathbf{S}^2 + 2\lambda\mu \mathbf{S} - \mu^2 \mathbf{I} \right] = 0. \tag{19}$$

Obviously, there is a vector $\mathbf{w}$ such that:

$$\left[\frac{1}{4}(1-\lambda)^2 \mathbf{r}\mathbf{r}^T - \lambda^2 \mathbf{S}^2 + 2\lambda\mu \mathbf{S} - \mu^2 \mathbf{I} \right] \mathbf{w} = 0. \tag{20}$$

This is an inhomogeneous $N \times N$ eigenproblem [7], and it can be reduced further to a $2N \times 2N$ standard eigenproblem by introducing the following quantity:

$$\mathbf{u} = \mu \mathbf{w}, \tag{21}$$

such that we have:

$$\left[\frac{1}{4}(1-\lambda)^2 \mathbf{r}\mathbf{r}^T - \lambda^2 \mathbf{S}^2 \right] \mathbf{w} + 2\lambda \mathbf{S}\mathbf{u} = \mu \mathbf{u}. \tag{22}$$

By combining the last two equations into a matrix representation we obtain:

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ A & B \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix} = \mu \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix}, \tag{23}$$

where