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Glued trees algorithm under phase damping

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ABSTRACT

We study the behaviour of the glued trees algorithm described by Childs et al. in [1] under decoherence. We consider a discrete time reformulation of the continuous time quantum walk protocol and apply a phase damping channel to the coin state, investigating the effect of such a mechanism on the probability of the walker appearing on the target vertex of the graph. We pay particular attention to any potential advantage coming from the use of weak decoherence for the spreading of the walk across the glued trees graph.

1. Introduction

One of the main difficulties for the grounding of a platform for quantum technologies is the effect of noise or "decoherence" on quantum states. No physical system is ever truly closed due to interactions with its environment. As a result of such interactions, the quantum state of the system will approach classicality, thus ceasing to be of interest for quantum-empowered protocols [2]. Before being able to create useful quantum technologies, we need to understand these processes, and eventually control them.

An intriguing aspect of decoherence is that, in specific cases (such as quantum stochastic resonance [3], to throw an example), weak decoherence mechanisms give rise to sizeable advantages in, say, the performance of some quantum protocols or the transport of excitations across a quantum medium. Such counterintuitive effects are tightly linked to quantum interference phenomena: decoherence changes the way the wave function of a given system evolves in time, thus affecting the occurrence of constructive and destructive interference. "Accidental" constructive effects may be induced, without spoiling the working principle of a given quantum process, for sufficiently weak decoherence mechanisms.

All this is particularly important (and evident) in the quantum walk framework [4], whose advantage in terms of the spreading rate of the position of a walker on a given "path" may be magnified by small degrees of phase noise. In this paper, we build on the already well established body of research into the behaviour of quantum walks when affected by decoherence, reported for instance in Refs. [5–7] and surveyed in Ref. [8].

After quickly revisiting the paradigm of quantum walks, we proceed to discuss the protocol under investigation, introducing phase noise and addressing the performance of the scheme for various strengths of such mechanism.

2. Quantum walks

A quantum walk is best described as the quantum analogue of the classical random walk. However, unlike the classical random walk, the evolution of a quantum walk is entirely deterministic. Quantum walks of course allow for superposition states of the walker, enabling them to exhibit interesting behaviours not shown by their classical counterparts. A comprehensive survey of quantum walks, covering both continuous and discrete time variants, and detailing the behaviours of quantum walks on various structures, can be found in Ref. [4]. In this paper, we focus on discrete time quantum walks.

A discrete time quantum walk operates within the Hilbert space $H = H_p \otimes H_c$, where H_p – known as the position space – describes the position of the walker on a well-defined structure (here we shall refer to this structure as the walk's *terrain*), and H_c – known as the coin space – describes an additional degree of freedom affecting the evolution of the walk: this degree of freedom determines the walker's behaviour in the next time step. For the evolution of the walk, we define two operators: the shift operator, S, and the coin operator, C. The shift operator will "move" the walker on to a new part of its terrain, depending on the coin state. For example, if the terrain of a walk is a graph, and the walker is on some vertex of the graph, the shift operator will move it along one of the vertex's edges to another vertex. The coin operator is analogous to the flipping of a coin in a classical random walk, it will act on the coin space which in turn affects how the walk shall

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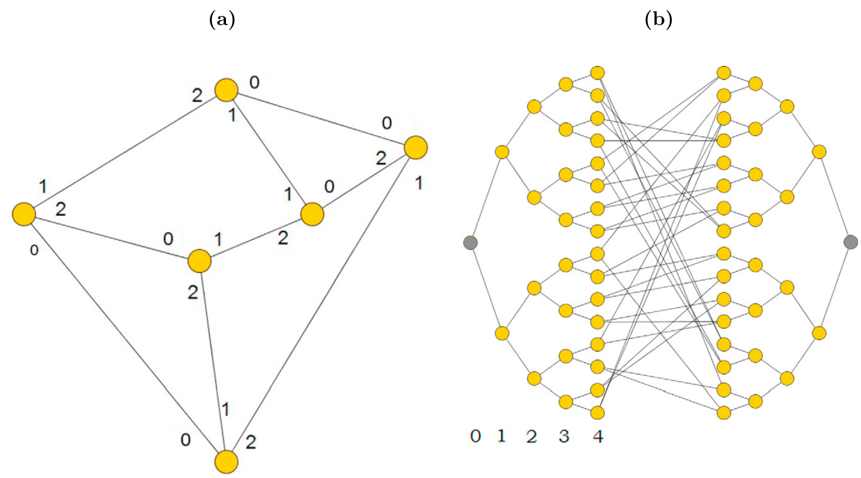

Fig. 1. (a) A labelled 3-regular graph. (b) A glued trees graph with 4 layers, $G'4$.

evolve in its terrain when the shift operator is applied. We move the walker along by one step by applying the coin operator followed by the shift operator; the state of the walker, starting in some initial state $|\psi(0)\rangle$, is thus described after t total steps by

$$\big|\psi(t)\big\rangle = \big[S(\mathbb{1}_p \otimes C)\big]^t \big|\psi(0)\big\rangle,$$

where $\mathbb{1}_p$ is the identity operator on the position space. In other words, we apply the coin operator and the shift operator t times to the initial walker state.

In order to provide a complete view of the main features of the walk protocol, we now give some concrete examples of quantum walks.

2.1. Discrete time quantum walk on a line

To represent the walk terrain, a line, we shall use the set of integers. The walker can be anywhere on the line, so we give H_p the basis $\{|i\rangle\colon\ i \in \mathbb{Z}\}$. As previously described, each step of the walk involves a coin flip and a shift. In the walk on the line, the walker has a "choice" of two directions, left and right. In the classical random walk, the decision of which direction to walk in at each step is reached by flipping a fair coin. Likewise, in our quantum walk we shall use a coin space of degree two, viz. H_c is given the basis $\{|0\rangle, |1\rangle\}$.

We decide to use the Hadamard operator for our coin, as in Ref. [4]. This has the effect of putting our walker into a superposition of coin states and will allow for interference to occur during the course of the evolution of the walk. With regards to the walker's behaviour on the terrain, the classical random walk will move one step to the left or one step to the right depending on the most recent coin flip. The same idea applies for the quantum walk. We define the shift operator as

$$S|p, c\rangle = \begin{cases} |p-1, c\rangle, & \text{if } c = 0, \\ |p+1, c\rangle, & \text{if } c = 1. \end{cases}$$

Ambainis et al. have shown in Ref. [9] that the quantum walk on the line spreads out quadratically faster than the classical random walk on the line.

2.2. Discrete time quantum walk on a k-regular graph

In general, a graph $G = (V, E)$ is specified by fixing a set of vertices V along with a set of edges E connecting them. k-regular graphs are graphs with k edges attached to each vertex. We affix a label $0 \leqslant l \leqslant k-1$ to each end of each edge, as illustrated in Fig. 1(a). The walker will traverse the graph's vertices, moving along the edges, so we define the position space H_p as having the basis $\{|p\rangle\colon\ p \in V\}$.

At each time step, the walker has a fan-out of k vertices to move to and we thus have to use an iso-dimensional coin space. We now give H_c the basis $\{|c\rangle\colon\ 0 \leqslant c \leqslant k-1\}$. We then introduce the Grover coin

$$C^{(G)}_{i,j} = \begin{cases} a, & \text{if } \delta_{i,j} = 1, \\ b, & \text{otherwise}, \end{cases} \tag{1}$$

which, as described in Ref. [4], generalises the Hadamard coin to Hilbert spaces of dimension larger than 2. In order for $C^{(G)}_{i,j}$ to be unitary, the conditions $|a|^2 + (k-1)|b|^2 = 1$ and $ab^* + a^*b + (k-2)|b|^2 = 0$ have to hold (with $a, b \in \mathbb{C}$). The values of a and b can be changed to vary the behaviour of the walk on the graph. We shall use a Grover coin later on to perform the simulations at the core of our work.

As for the shift operator, this must take the walker along the appropriate edge to a new vertex, depending on the coin state. Again, we state that this idea is a generalisation of the walk on the line in which we give the walker a choice of k directions at each step rather than 2. We define our shift operator as

$$S|v, c\rangle = \big|w, c'\big\rangle, \tag{2}$$

where $(v, w) \in G$ and is labelled c on v's end, and c' is the label assigned to the destination node's end of the edge.

3. Model used

The goal of the glued trees (GT) algorithm for quantum search is the following: beginning from the left-most vertex of a given GT graph, traverse the graph and reach the right-most vertex, referred to as the target vertex. Childs et al. [1] use this algorithm to show quantum walk search to be fundamentally more effective than classical random walk search by presenting a class of graphs (the GT graphs) that force classical random walks to make exponentially many queries to an oracle encoding the structure of the graph, but that are traversable by quantum walks with a polynomial number of queries to such an oracle. In order to study the robustness of the algorithm to the detrimental effects of decoherence, we shall determine how effectively it achieves its goal when subjected to an increasing degree of phase damping noise. For this reason, we will focus on the probability that the walker is on the target vertex at the end of the walk. We thus consider GT graphs such as the one illustrated in Fig. 1(b), i.e. consisting of n layers before the gluing stage, and thus labelled as $G'n$.