Physics Letters A 378 (2014) 367–373

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

# Necessity of $\mathcal{PT}$ symmetry for soliton families in one-dimensional complex potentials

Jianke Yang *

*Department of Mathematics and Statistics, University of Vermont, Burlington, VT 05401, USA*

ARTICLE INFO

*Article history:*
Received 16 October 2013
Accepted 26 November 2013
Available online 12 December 2013
Communicated by A.R. Bishop

*Keywords:*
$\mathcal{PT}$ symmetry
Soliton families
NLS equations

ABSTRACT

For the one-dimensional nonlinear Schrödinger equation with a complex potential, it is shown that if this potential is not parity-time ($\mathcal{PT}$) symmetric, then no continuous families of solitons can bifurcate out from linear guided modes, even if the linear spectrum of this potential is all real. Both localized and periodic non-$\mathcal{PT}$-symmetric potentials are considered, and the analytical conclusion is corroborated by explicit examples. Based on this result, it is argued that $\mathcal{PT}$-symmetry of a one-dimensional complex potential is a necessary condition for the existence of soliton families.



## 1. Introduction

Nonlinear wave systems used to be divided into two main categories: conservative systems and dissipative systems. In the former category, the system has no energy gain or loss, and solitary waves (or solitons in short) exist as continuous families, parameterized by their propagation constants. A well known example is the nonlinear Schrödinger equation with or without a real potential [1,2]. In the latter category, the system has energy gain and loss, and solitons generally exist as isolated solutions at certain discrete propagation-constant values, where the energy gain and loss on the soliton are exactly balanced (such solitons are often referred to as dissipative solitons in the literature) [3]. A typical example in this latter category is the Ginzburg–Landau equation or short-pulse lasers (see also [4,5]).

However, a recent discovery is that, in dissipative but parity-time ($\mathcal{PT}$) symmetric systems [6], solitons can still exist as continuous families, parameterized by their propagation constants [7–24]. This exact balance of continually deformed wave profiles in the presence of gain and loss is very remarkable.

The existence of soliton families in conservative and $\mathcal{PT}$-symmetric systems can be intuitively understood as follows. In both cases, the linear spectrum of the system is all-real or partly-real [6,25]. This means that the system supports linear guided modes. Then under nonlinearity, these linear guided modes can bifurcate out, leading to continuous families of solitons. In typical dissipative systems (such as the Ginzburg–Landau equation), however, the linear spectrum is all complex. Since there are no linear guided modes, soliton-family bifurcation from linear modes then is not possible. As a result, it is understandable that only isolated solitons can exist in such typical dissipative systems.

It turns out that non-$\mathcal{PT}$-symmetric dissipative systems can also possess all-real or partly-real linear spectra. Indeed, for the one-dimensional (1D) linear Schrödinger operator, various non-$\mathcal{PT}$-symmetric complex potentials with all-real spectra have been constructed by the supersymmetry method [26–28]. Other non-$\mathcal{PT}$-symmetric dissipative systems with partly-real spectra have been reported as well [4,29,30]. In such non-$\mathcal{PT}$-symmetric dissipative systems, since linear guided modes exist, then an important question is: can continuous families of solitons bifurcate out from them? If they do, then the underlying non-$\mathcal{PT}$-symmetric dissipative system would allow much more flexibility in steering nonlinear localized modes (such as optical solitons) with continuous ranges of intensities, and this flexibility could have potential physical applications.

In this article, we investigate the existence of soliton families in the 1D nonlinear Schrödinger (NLS) equation with a non-$\mathcal{PT}$-symmetric complex potential. This NLS system governs paraxial nonlinear light propagation in a medium with non-$\mathcal{PT}$-symmetric refractive-index and gain-loss landscape [2,7], as well as Bose–Einstein condensates with a non-$\mathcal{PT}$-symmetric trap and gain-loss distribution [31]. In this NLS model, we show that no soliton families can bifurcate out from localized linear modes of a non-periodic potential or Bloch-band edges of a periodic potential. This means that no soliton families can bifurcate out from linear guided modes (if such modes exist). This result suggests that 1D non-$\mathcal{PT}$-symmetric potentials do not support continuous families of solitons. In other words, $\mathcal{PT}$-symmetry of a 1D complex potential is a necessary condition for the existence

* Tel.: +1 802 6564314; fax: +1 802 6562552.
*E-mail address:* jyang@math.uvm.edu.

0375-9601/$ – see front matter 
http://dx.doi.org/10.1016/j.physleta.2013.11.033

of soliton families (although it is not necessary for all-real linear spectra).

## 2. Preliminaries

The model equation we consider is the following 1D NLS equation with a linear non-$\mathcal{PT}$-symmetric complex potential

$$iU_t + U_{xx} - V(x)\Psi + \sigma|\Psi|^2\Psi = 0, \tag{2.1}$$

where $V(x)$ is complex-valued and non-$\mathcal{PT}$-symmetric, i.e.,

$$V^*(x) \neq V(-x), \tag{2.2}$$

the asterisk represents complex conjugation, and $\sigma = \pm 1$ is the sign of nonlinearity. This equation governs paraxial light transmission as well as Bose–Einstein condensates in non-$\mathcal{PT}$-symmetric media. In this model, the nonlinearity is cubic. But extension of our analysis to an arbitrary form of nonlinearity is straightforward without much more effort [32].

Regarding the non-$\mathcal{PT}$-symmetric potential $V(x)$, a remark is in order. If this $V(x)$ is non-$\mathcal{PT}$-symmetric, but becomes $\mathcal{PT}$-symmetric after a certain spatial translation $x_0$, i.e., $V(x - x_0)$ is $\mathcal{PT}$-symmetric, then wave dynamics in this non-$\mathcal{PT}$-symmetric potential $V(x)$ is equivalent to that in the $\mathcal{PT}$-symmetric potential $V(x - x_0)$ and is thus not the subject of our study. Hence, in this article we require that the non-$\mathcal{PT}$-symmetric potential $V(x)$ in Eq. (2.1) remains non-$\mathcal{PT}$-symmetric under any spatial translation.

For non-$\mathcal{PT}$-symmetric complex potentials, their linear spectra may or may not contain real eigenvalues. In this article, we will consider those potentials that admit real eigenvalues in their linear spectra. Non-$\mathcal{PT}$potentials with all-real spectra are special but important examples of such potentials.

We seek solitons in Eq. (2.1) of the form

$$U(x,t) = e^{i\mu t}u(x), \tag{2.3}$$

where $u(x)$ is a localized function satisfying the equation

$$u_{xx} - V(x)u - \mu u + \sigma|u|^2u = 0, \tag{2.4}$$

and $\mu$ is a real-valued propagation constant. The question we will investigate is, does this equation admit soliton families for a continuous range of propagation-constant values when the potential $V(x)$ is non-$\mathcal{PT}$-symmetric?

It is noted that Eq. (2.4) is phase-invariant. That is, if $u(x)$ is a solitary wave, then so is $u(x)e^{i\alpha}$, where $\alpha$ is any real constant. In this article, solitons that are related by this phase invariance will be considered as equivalent.

## 3. Non-existence of soliton families bifurcating from localized linear modes

In this section, we consider non-$\mathcal{PT}$-symmetric potentials that are not periodic (for instance, localized potentials). Such potentials can admit discrete real eigenvalues, i.e., linear guided modes [4, 26–30]. If this potential were real or $\mathcal{PT}$-symmetric, soliton families would always bifurcate out from those linear guided modes (see the last section of this article). However, when the potential is non-$\mathcal{PT}$-symmetric, we will show that such soliton-family bifurcations are forbidden.

Suppose $V(x)$ is a non-$\mathcal{PT}$-symmetric potential which admits a simple discrete real eigenvalue $\mu_0$, with the corresponding localized eigenfunction $\psi(x)$, i.e.,

$$L\psi = 0, \tag{3.1}$$

where

$$L \equiv \frac{d^2}{dx^2} - V(x) - \mu_0. \tag{3.2}$$

Since $\mu_0$ is a simple eigenvalue, the equation $L\psi_g = \psi$ for the generalized eigenfunction $\psi_g$ should not admit any solution. This means that the solvability condition of this $\psi_g$ equation should not be satisfied, i.e., its inhomogeneous term $\psi$ should not be orthogonal to the adjoint homogeneous solution $\psi^*$, or

$$\langle\psi^*, \psi\rangle \neq 0, \tag{3.3}$$

where

$$\langle f, g\rangle \equiv \int_{-\infty}^{\infty} f^*(x)g(x)\,dx \tag{3.4}$$

is the standard inner product.

If a soliton family in Eq. (2.4) bifurcates out from this localized linear eigenmode, then we can expand these solitons into a perturbation series. We will show that this perturbation series requires an infinite number of nontrivial conditions to be satisfied simultaneously, which is impossible in practice due to lack of spatial symmetries in the 1D potential $V(x)$.

To proceed, let us expand these solitons into a perturbation series

$$u(x;\mu) = \epsilon^{1/2}\left[u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \cdots\right], \tag{3.5}$$

where $\epsilon \equiv \mu - \mu_0$ is small. Substituting this expansion into Eq. (2.4), at $O(\epsilon^{1/2})$ we get

$$Lu_0 = 0, \tag{3.6}$$

hence

$$u_0 = c_0\psi, \tag{3.7}$$

where $c_0$ is a certain non-zero constant.

At $O(\epsilon^{3/2})$, we get the equation for $u_1$ as

$$Lu_1 = c_0\left(\psi - \sigma|c_0|^2|\psi|^2\psi\right). \tag{3.8}$$

Here the $u_0$ solution (3.7) has been utilized. The solvability condition of this $u_1$ equation is that its right hand side be orthogonal to the adjoint homogeneous solution $\psi^*$. This condition yields an equation for $c_0$ as

$$|c_0|^2 = \frac{\langle\psi^*, \psi\rangle}{\sigma\langle\psi^*, |\psi|^2\psi\rangle}. \tag{3.9}$$

Here we have assumed that the denominator $\langle\psi^*, |\psi|^2\psi\rangle \neq 0$. If it is zero, perturbation expansions different from (3.5) would be needed, but the qualitative result would remain the same as that given below.

Since $|c_0|$ is real and $\sigma = \pm 1$, Eq. (3.9) then requires that

$$Q_1 \equiv \frac{\langle\psi^*, \psi\rangle}{\langle\psi^*, |\psi|^2\psi\rangle} \quad \text{must be real.} \tag{3.10}$$

In a non-$\mathcal{PT}$-symmetric complex potential, $Q_1$ is generically complex, thus this condition is generically not satisfied.

It turns out that Eq. (3.10) is only the first condition for soliton-family bifurcations. As we pursue the perturbation expansion (3.5) to higher orders, infinitely more conditions will also appear. This will be demonstrated below.

If condition (3.10) is met, then the $u_1$ equation (3.8) is solvable. Its solution is

$$u_1 = \hat{u}_1 + c_1\psi, \tag{3.11}$$