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Mechanical analogues of spin Hamiltonians and dynamics

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ABSTRACT

Bloch et al. mapped the precession of the spin-half in a magnetic field of variable magnitude and direction to the rotations of a rigid sphere rolling on a curved surface utilizing $SU(2)$–$SO(3)$ isomorphism. This formalism is extended to study the behaviour of spin–orbit interactions and the mechanical analogy for Rashba–Dresselhauss spin–orbit interaction in two dimensions is presented by making its spin states isomorphic to the rotations of a rigid sphere rolling on a ring. The change in phase of spin is represented by the angle of rotation of sphere after a complete revolution. In order to develop the mechanical analogy for the spin filter, we find that perfect spin filtration of down spin makes the sphere to rotate at some unique angles and the perfect spin filtration of up spin causes the rotations with certain discrete frequencies.

Spin is a purely quantum mechanical concept with no analogue in classical mechanics. Thus, the description of classical aspects of a quantum mechanical phenomenon like spin–orbit interactions becomes a complex issue. Spin–orbit interactions play an important role in understanding atomic and nuclear structure. Littlejohn and Flynn employed multicomponent wavefunctions to WKB quantization of integrable spherical spin–orbit coupled systems [1]. The semiclassical description breaks down at those points (or subspaces) of classical phase space where the spin–orbit interaction locally becomes zero. However, Frisk and Guhr have investigated spin–orbit interactions in non-spherical potentials to study the shell structure in atomic nuclei and resolved the problem of mode conversion in the case of planar orbits [2]. Bolte and Keppeler derived the relativistic trace formula for Dirac equation by following the technique developed by Gutzwiller for Schrödinger's equation [3]. By a path integral approach, it was Klauder, who gave the formulation for a system with spin in $SU(2)$ spin coherent state representation as an integral over the sphere S^2 [4]. Subsequently, Kuratsuji et al. represented the path integral in $SU(2)$ spin coherent state as an integral over the paths in extended complex plane $\bar{C}^1$ [5,6]. But the exact form of $SU(2)$ coherent state path integral representation for transition amplitude, involving the boundary term and appropriate boundary conditions, was developed by Kochetov [7]. Adding to his formalism, Pletyukhov et al. [8] calculated the ingredients of Gutzwiller's trace formula for the density of states and tested it for a two-dimensional quantum dot with a spin–orbit interaction of Rashba type.

We connect the quantum-mechanical phenomenon of spin–orbit interactions to its classical analogue by utilizing $SU(2)-SO(3)$ isomorphism. Bloch et al. discussed the precession of a spin-half in an external magnetic field by mapping $SU(2)$ spin to $SO(3)$ rotations of a rigid sphere rolling on a curved surface [9]. Their formulation is extended and a spinor undergoing spin–orbit interaction is mapped to rigid sphere rolling on a ring. While deriving the formulations for an analogous picture of the eigenstates of the spin Hamiltonian, no approximations are made. Hence, the exact solutions obtained by this formalism provide a classical description of spin–orbit interactions. The dynamics of this model is also very interesting. The trajectories painted on the sphere rolling on the curved surface are actually the instantaneous measure of the magnetic moment associated with a spin-half particle. In Section 1, we present the basic ideas, following [9], and then we allow ourselves a leap of imagination by exploiting the mathematical similarity of $\mathbf{B}$ and $\mathbf{L}$ as axial vectors.

1. Motivation

Consider a rigid sphere rolling along a curve $\boldsymbol{\Gamma}$ on a plane. An inertial coordinate system called the spatial coordinate system with its origin at the center of the sphere is fixed. The position and the instantaneous velocity with respect to the center of the sphere of a given particle in the body at time t are denoted by $\mathbf{X}(t)$ and $\dot{\mathbf{X}}(t)$. At each instant t, there exists a unique angular velocity $\boldsymbol{\omega}$ such that, for every particle in the body,

$$\dot{\mathbf{X}}(t) = \boldsymbol{\omega} \times \mathbf{X}(t). \tag{1}$$

The translational velocity of the rolling sphere is

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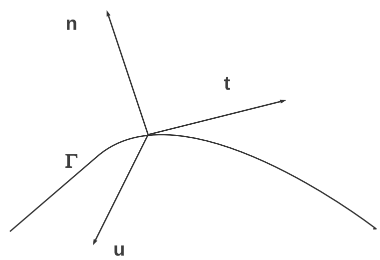

Fig. 1. Illustration of directions **n** (normal vector), **t** (tangent vector) and **u** to the curve $\boldsymbol{\Gamma}$.

$$\mathbf{V} = \mathbf{t}V(t) = \mathbf{t}\frac{ds}{dt},$$

where s is the arc length of the curve $\boldsymbol{\Gamma}$ and $\mathbf{t}$ is the tangent to the curve $\boldsymbol{\Gamma}$ [10]. It is well known that the rolling constraint means that the instantaneous velocity at the point of contact is zero i.e.

$$\boldsymbol{\omega} \times \mathbf{n}R = \mathbf{t}\frac{ds}{dt} \tag{2}$$

where R is the radius of the sphere, $\mathbf{n}$ is the normal to the surface. Now the expression for $\boldsymbol{\omega}$ can be obtained by taking the cross product of expression (2) with $\mathbf{n}$, i.e.

$$\boldsymbol{\omega} = \frac{V(t)}{R}\mathbf{n} \times \mathbf{t} = \frac{1}{R}\frac{ds}{dt}\mathbf{u} \tag{3}$$

where $\mathbf{u} = \mathbf{n} \times \mathbf{t}$ is the tangent normal as described in Fig. 1 and the rolling without instantaneous rotation about the normal, i.e. $\boldsymbol{\omega}.\mathbf{n} = 0$ is considered. Rewriting Eq. (1) by using the expression (3) for $\boldsymbol{\omega}$, we obtain

$$\frac{d\mathbf{X}}{ds} = \frac{\mathbf{u} \times \mathbf{X}}{R}. \tag{4}$$

If we compare this equation with Larmor precession of the magnetic moment in the external magnetic field as it exerts a torque $\boldsymbol{\tau}$ on the magnetic moment

$$\boldsymbol{\tau} = -\mathbf{B} \times \frac{\boldsymbol{\mu}}{\gamma} = -\mathbf{B} \times \mathbf{J},$$

where $\gamma = \frac{e}{2m}$ is the gyromagnetic ratio, then $\mathbf{X} = (x, y, z)$ is identified as magnetic dipole moment and s as time. Eq. (3) describes precession of the angular momentum vector $\mathbf{J}$ with frequency

$$\mathbf{B} = -\frac{(u_x, u_y, u_z)}{R} = -\boldsymbol{\omega_s}$$

of constant magnitude $\frac{1}{R}$ and direction $(-\mathbf{u})$ varying with s. There is an isomorphism between the rolling sphere written in this way with a spin-half precessing in the magnetic field. This can be seen if (using $\mathbf{B} = -\boldsymbol{\omega_s}$) (4) is rewritten in the form

$$\frac{d}{ds}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{5}$$

which is same as the following equations of motion for two complex numbers a and b

$$i\frac{d}{ds}\begin{bmatrix} a \\ b \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}, \tag{6}$$

where (x, y, z) and (a, b) are related via

$$x = \left(ab^* + ba^*\right),$$
$$y = i\left(ab^* - ba^*\right),$$
$$z = \left(aa^* - bb^*\right), \quad \text{and}$$
$$x^2 + y^2 + z^2 = 1. \tag{7}$$

From Eq. (7), the behaviour of rolling sphere as a function of arc length, which we are taking as equivalent to time, can be determined. The real numbers (x, y, z) represent the coordinates of a point within the sphere at a unit distance from the center in the spatial coordinate system with its origin at the center of the sphere. Thus, the precession of a spin-half in a magnetic field of variable magnitude and direction is mapped to the rotations of a rigid sphere on a curved surface as the (x, y, z) coordinates have already been identified as component of the magnetic moment. Eq. (6) – the equations of motion for the rolling sphere on a curved surface are made equivalent to the Schrödinger's equation for the spinor $\boldsymbol{\chi}^T = (a, b)$ in the presence of a magnetic field $\mathbf{B}$ because of $SU(2)$–$SO(3)$ isomorphism,

$$i\frac{d}{ds}\chi = -\mathbf{B}.\mathbf{S}\chi \tag{8}$$

where $\hbar = 1$, $\mathbf{S} = \frac{1}{2}(\sigma_x, \sigma_y, \sigma_z)$ is the spin operator and $(\sigma_x, \sigma_y, \sigma_z)$ are Pauli spin matrices [9]. It is well known that the magnetic field is associated with the angular momentum of the particle by the following relation:

$$\mathbf{B} = \frac{1}{r'}\frac{\partial V_{so}}{\partial r'}\mathbf{L}, \quad \text{where } \mathbf{L} = \mathbf{r}' \times \mathbf{p}'$$

is the angular momentum and V_{so} is the potential energy associated with the spin–orbit interactions in the central field. So, Eq. (8) is now written as

$$i\frac{d}{ds}\chi = -\frac{1}{r'}\frac{\partial V_{so}}{\partial r'}\mathbf{L}.\mathbf{S}\chi = -k\mathbf{L}.\mathbf{S}\chi \tag{9}$$

where $k = \frac{1}{r'}\frac{\partial V_{so}}{\partial r'}$ is the strength of spin–orbit interaction and the Hamiltonian of the system undergoing spin–orbit interaction is given as $\mathbf{H} = -k\mathbf{L}.\mathbf{S}$.

Thus, the behaviour of the spin–orbit interactions can be studied by mapping them to the rotations of the sphere rolling on a curved surface. In [9], precessing of spin about a magnetic field is considered whereas we are considering spin interacting with the orbital angular momentum $\mathbf{L}$. Due to fact that mathematics is similar, we succeed, but the thought of extending it thus is non-trivial.

Rashba and Dresselhaus spin–orbit interactions are chosen as special case since they play significant role in dephasing of the spin components in spintronic devices. The mechanical analogy of these interaction Hamiltonian is developed by identifying its isomorphism with a rigid sphere rolling on a ring. A device called, spin filter, allows us to choose only one component of the spin. We develop the mechanical analogue for this device and study how the perfect filtration of each component of spin is related to the rotations of the sphere.

2. Rashba and Dresselhaus spin–orbit interaction Hamiltonian

In two dimensional III–V semiconductor systems, there are two distinct Hamiltonian terms contributing to spin dephasing – "bulk inversion asymmetry" term and "structure inversion asymmetry" term. These appear only in asymmetric systems. The bulk inversion asymmetry term arises from Dresselhaus spin-splitting while the structure inversion symmetry arises from Rashba spin splitting [11]. The coupling constant in the case of Rashba spin–orbit interaction Hamiltonian is proportional to the external magnetic field but in case of Dresselhaus interaction Hamiltonian, the coupling constant is proportional to the crystal field.

The Hamiltonian for the spin–orbit interaction of Rashba type is given as:

$$H = k[\sigma_y p_x - \sigma_x p_y] = k\left[\boldsymbol{\sigma}.(\hat{\mathbf{z}} \times \mathbf{p})\right] = 2k\left[(\hat{\mathbf{z}} \times \mathbf{p}).\mathbf{S}\right]$$

where k is the spin–orbit coupling strength. Hence, Schrödinger equation for the spinor $\boldsymbol{\psi}^T = (a, b)$, taking s as time, is given as: