Physics Letters A 377 (2013) 2284–2287

Contents lists available at SciVerse ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Fisher information, nonclassicality and quantum revivals

Elvira Romera [a,b], Francisco de los Santos [a,c,*]

[a] Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, Fuentenueva s/n, 18071 Granada, Spain
[b] Departamento de Física Atómica, Molecular y Nuclear, Universidad de Granada, Fuentenueva s/n, 18071 Granada, Spain
[c] Departamento de Electromagnetismo y Física de la Materia, Universidad de Granada, Fuentenueva s/n, 18071 Granada, Spain

ARTICLE INFO

Article history:
Received 16 May 2013
Received in revised form 21 June 2013
Accepted 24 June 2013
Available online 1 July 2013
Communicated by P.R. Holland

ABSTRACT

Wave packet revivals and fractional revivals are studied by means of a measure of nonclassicality based on the Fisher information. In particular, we show that the spreading and the regeneration of initially Gaussian wave packets in a quantum bouncer and in the infinite square-well correspond, respectively, to high and low nonclassicality values. This result is in accordance with the physical expectations that at a quantum revival wave packets almost recover their initial shape and the classical motion revives temporarily afterward.

1. Introduction

Quantum systems may behave classically or quasiclassically under a variety of circumstances and, in this regard, the transition from quantum to classical mechanics still poses intriguing problems that attract considerable attention. Of particular interest are systems that display both classical and quantum periodic motions, with generally different, incommensurable periods, for in this case the interesting question arises as to how the classical periodicity emerges from the quantum one in the appropriate limit. For instance, a particle of mass m and energy E in an infinite square-well potential of width L initially oscillates with a classical period $T_{\text{cl}} = L\sqrt{2m/E}$. The classical oscillations gradually damp out as the wave packet representing the particle spreads more or less uniformly across the well. Quantum mechanically, the wave function regains exactly its initial form with a *revival* period $T_{\text{rev}} = 4mL^2/\pi\hbar$, after which the classical oscillations resume with period T_{cl} again. At times that are rational fractions of T_{rev}, the wave packet temporarily splits into a number of scaled copies called fractional revivals [1–3]. In the same vein, an object of mass m released from a height z_0 and subjected only to gravity, bounces up and down against an impenetrable flat surface with a classical period (in suitable units) $T_{\text{cl}} = 2\sqrt{z_0}$, while the wave function of the corresponding quantum bouncer *almost* returns to its initial shape after a revival time $T_{\text{rev}} = 4z_0^2/\pi$. After a revival has taken place, a new cycle of quasiclassical behavior and revivals commences again. The fact that in this case the revivals are only approximate does not make a difference. Revivals and fractional revivals received a great deal of attention over the last decades. Theoretical progress and experimental observations were made in atoms and molecules, and Bose–Einstein condensates [4–7]. Recently, revivals have been theoretically investigated in low dimensional systems [8–13] and have been related to quantum phase transitions [14].

Identifying the occurrence of wave packet revivals usually makes use of the autocorrelation function $A(t) = \langle\Psi(0)|\Psi(t)\rangle$, which is the overlap between the initial and the time-evolving wave packet. Within this approach, the occurrence of revivals and fractional revivals corresponds to, respectively, the return of $A(t)$ to its initial value of unity and the appearance of relative maxima in $A(t)$. Another method to study revival phenomena consists in monitoring the time evolution of the expectation values of some quantities [3,15,16], and an approach based on a finite difference eigenvalue method has been put forward that allows to predict the revival times directly [17]. Recently, information entropy approaches were proposed [18] based on the Shannon and Rényi entropies, complementary to the conventional ones. This technique was shown to be superior to analyses based on both the standard variance uncertainty product [19] and the autocorrelation function, inasmuch as it overcomes the difficulty that wave packets reform themselves at locations that do not coincide with their original ones. A complementary informational measure is the Fisher information [20] which has attracted substantial interest in physics, in particular in atomic and molecular physics (see for example [21–35]). In this Letter we show that the analysis of the wave packet dynamics can be carried out using a new tool, namely, the *nonclassicality* J_{nc}, defined in terms of the Fisher information as we now discuss.

Hall [36] has recently introduced a measure of nonclassicality, J_{nc}, in terms of the probability densities in position and momentum spaces, $\rho(x) = |\psi(x)|^2$ and $\gamma(p) = |\phi(p)|^2$, respectively. To be

* Corresponding author at: Departamento de Electromagnetismo y Física de la Materia, Universidad de Granada, Fuentenueva s/n, 18071 Granada, Spain. Tel.: +34 958 244 014.
E-mail address: dlsantos@onsager.ugr.es (F. de los Santos).

0375-9601/$ – see front matter
http://dx.doi.org/10.1016/j.physleta.2013.06.037

concrete, $J_{\rm nc} \equiv (\hbar/2)\sqrt{I_\rho I_\gamma}$, where

$$I_\rho = 4\int \left|\frac{d}{dx}\rho^{1/2}(x)\right|^2 dx, \qquad I_\gamma = 4\int \left|\frac{d}{dp}\gamma^{1/2}(p)\right|^2 dp. \tag{1}$$

Note that I_ρ and I_γ are the classical Fisher informations associated with the probability densities $\rho(x)$ and $\gamma(p)$ [20]. In the next section we shall show that the time evolution of a wave packet exhibiting revivals and fractional revivals is initially characterized by a classical behavior with a low $J_{\rm nc}$ value, followed by a wave packet spreading with a higher value of $J_{\rm nc}$. In the long-time evolution, for times near $T_{\rm rev}$, the wave packet (approximately) restores its initial form, exhibiting classical periodicity again accompanied by a decrease in $J_{\rm nc}$. In the case of fractional revivals at $t = pT_{\rm rev}/q$, several mini-packets emerge whereupon a decrease in $J_{\rm nc}$ is expected.

This Letter is organized as follows. In Section 2 we shall consider the Fisher information as it applies to revival phenomena. In particular, we show the role of the Fisher information as a measure of nonclassicality in the dynamics of two model systems that exhibit fractional revivals: the so-called quantum 'bouncer', that is a quantum particle bouncing against a hard surface under the influence of gravity and the infinite square-well. Finally, some concluding remarks will be given in the last section.

2. Fisher information and nonclassicality

The Fisher information of single particle systems is defined as a functional of the density function in conjugate spaces by Eq. (1) and it has been shown to be a measure of nonclassicality [36]. Following Hall [36] and Mosna et al. [37], the Fisher information in position space can be expressed as $I_\rho = (4/\hbar)^2(\langle P^2\rangle_\psi - \langle P_{\rm cl}^2\rangle_\psi)$ where P denotes the momentum operator and $P_{\rm cl}$ is a (state-dependent) classical momentum operator defined by

$$P_{\rm cl}\Psi(x) = \frac{\hbar}{2i}\left(\frac{\psi'(x)}{\psi(x)} - \frac{\psi^{*\prime}(x)}{\psi^*(x)}\right) = \hbar\big(\arg\psi(x)\big)'. \tag{2}$$

Hence, it is natural to separate the momentum operator in a classical ($P_{\rm cl}$) and nonclassical ($P_{\rm nc}$) contribution with $P_{\rm nc} \equiv P - P_{\rm cl}$. The definition of the classical momentum observable is supported by the facts that ρ satisfies the classical continuity equation and that the expectation values of P and $P_{\rm cl}$ are equal for all wave functions $\langle P\rangle_\psi = \langle P_{\rm cl}\rangle_\psi$ [36]. The conjugate equality that relates the momentum Fisher information and the nonclassicality of the position operator can be obtained analogously, $I_\gamma = (4/\hbar)^2(\langle X^2\rangle_\phi - \langle X_{\rm cl}^2\rangle_\phi)$. Finally, Hall introduced a measure of joint nonclassicality $J_{\rm nc}$ for a quantum state as

$$J_{\rm nc} \equiv \frac{\hbar}{2} I_\rho^{1/2} I_\gamma^{1/2}. \tag{3}$$

It follows that $J_{\rm nc} = 1$ for Gaussian distributions. For instance, the evolution of Gaussian wave packets in a harmonic oscillator follows a periodic motion in accordance with classical expectations [3], and $J_{\rm nc} = 1$ for all times. For mixed states, $J_{\rm nc}$ can be arbitrarily small while for pure states Hall found $J_{\rm nc} \geqslant |1 + (i/\hbar)\langle[P_{\rm cl}, X_{\rm cl}]\rangle_\psi|$ [36].

2.1. Quantum bouncer

Consider an object of mass m bouncing against a hard surface subjected only to the influence of the gravitational force directed downward along the z axis, that is, a particle in a potential $V(z) = mgz$, if $z > 0$ and $V(z) = +\infty$ otherwise. Gravitational quantum bouncers have been recently realized using neutrons [38] and atomic clouds [39]. Their revival behavior has been discussed in [16,40] and an entropy-based approach was presented in [18,19].

The time-dependent wave function for a localized quantum wave packet is expanded as a one-dimensional superposition of energy eigenstates as

$$\psi(x,t) = \sum_n a_n u_n(x) e^{-iE_n t/\hbar}. \tag{4}$$

The eigenfunctions and eigenvalues are given by [40]

$$E_n' = z_n; \qquad u_n(z') = \mathcal{N}_n \,\mathrm{Ai}(z' - z_n); \quad n = 1, 2, 3, \ldots \tag{5}$$

where $l_g = (\hbar/2gm^2)^{1/3}$ is a characteristic gravitational length, $z' = z/l_g$, $E' = E/mgl_g$, $\mathrm{Ai}(z)$ is the Airy function, $-z_n$ denotes its zeros, and $\mathcal{N}_n = |\mathrm{Ai}'(-z_n)|$ is the $u_n(z')$ normalization factor. Accurate analytic approximations for z_n exist [40],

$$z_n \simeq \frac{3\pi}{2}\left[n - \frac{1}{4}\right]^{2/3}. \tag{6}$$

Consider now an initial Gaussian wave packet localized at a height z_0 above the surface, with a width σ and a momentum p_0 (in the remainder of this Letter the primes on the energy and position symbols will be dropped)

$$\psi(z,0) = \frac{1}{\sqrt{\sigma\hbar\sqrt{\pi}}} e^{-(z-z_0)^2/2\sigma^2\hbar^2} e^{ip_0(z-z_0)/\hbar}. \tag{7}$$

If the lower bound of the integral is extended to $-\infty$, the associated coefficients of the time-dependent wave function for $p_0 = 0$ can be obtained analytically as [40]

$$\begin{aligned} C_n &\simeq \mathcal{N}_n\left(\frac{2}{\pi\sigma^2}\right)^{1/4}\int_{-\infty}^{\infty} \mathrm{Ai}(z - z_n) e^{-(z-z_0)^2/\sigma^2}\, dz \\ &= \mathcal{N}_n\left(\frac{2}{\pi\sigma^2}\right)^{1/4}\sqrt{\pi}\sigma \exp\left[\frac{\sigma^2}{4}\left(z_0 - z_n + \frac{\sigma^4}{24}\right)\right] \\ &\quad\times \mathrm{Ai}\left(z_0 - z_n + \frac{\sigma^4}{16}\right). \end{aligned} \tag{8}$$

The important time scales of a wave packet's time evolution are in the coefficients of the Taylor series (see, for instance [1–3]) of the energy spectrum E_n around the level the wave packet is centered around, let us say $\bar{n}$:

$$E_{\bar{n}} = E_n + 2\pi\hbar\left(\frac{(n-\bar{n})}{T_{\rm cl}} + \frac{(n-\bar{n})^2}{T_{\rm rev}} + \cdots\right). \tag{9}$$

The classical period and the revival time can be calculated to obtain $T_{\rm cl} = 2\sqrt{z_0}$ and $T_{\rm rev} = 4z_0^2/\pi$, respectively [40]. The temporal evolution of the wave packet in momentum space was obtained numerically by the fast Fourier transform method.

We have computed the temporal evolution of the autocorrelation function and the nonclassicality $J_{\rm nc}$ for the initial conditions $z_0 = 100$, $\sigma = 1$ and $p_0 = 0$. Fig. 1 displays the early time evolution of both quantities and the location of the main fractional revivals. The top panel shows how the autocorrelation function initially follows the first classical periods of motion. The nonclassicality, $J_{\rm nc}$, describes precisely this same behavior, with peaks at the wave packet's collapses and minima at the multiples of the classical period. In the long-time limit, the wave packet eventually spreads out and collapses, only to reform at multiples of the revival time. In between, fractional revivals take place. All this is reflected in the maxima and relative maxima of $|A(t)|^2$ as shown in the top panel of Fig. 2. The alternative description in terms of $J_{\rm nc}$ is shown in the bottom panel of Fig. 2. The slightly nonclassical behavior or, equivalently, the quasiclassical behavior that takes place at full and