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A variational approach to the modulational-oscillatory instability of Bose–Einstein condensates in an optical potential

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ABSTRACT

We use the time-dependent variational approach to demonstrate how the modulational and oscillatory instabilities can be generated in Bose–Einstein condensates (BECs) trapped in a periodic optical lattice with *weak* driving harmonic potential. We derive and analyze the ordinary differential equations for the time evolution of the amplitude and phase of the modulational perturbation, and obtain the instability condition of the condensates through the effective potential. The effect of the optical potential on the dynamics of the BECs is shown. We perform direct numerical simulations to support our theoretical findings, and good agreement is found.

1. Introduction

Inspired by the work of Bose and Einstein in 1925, the first experimental demonstration of Bose–Einstein condensates (BECs) was carried out with dilute alkali gases recently in 1995 [1–3]. This achievement has then triggered numerous research works in the field of ultracold atoms. Among them the experimental [4–6] and theoretical [7–17] investigations of the dynamics and properties of BECs in optical lattice (OL) potentials have been a central topic. In these experiments, the OL is created by two counterpropagating laser beams forming a standing wave interference pattern. The dynamic properties of the atoms are characterized by the depth and the period of this optically-induced potential. The intensity of the OL potential can be modulated from very weak to very strong [4]. Hence, BECs in periodic potentials have been found useful in investigating many physical phenomena such as Josephson effect [18], Bloch oscillations [4,19,20], Landau–Zener tunneling [21], solitons [22], quantum phase transitions of the Mott insulator type [23], superfluid and dissipative dynamics [24], phase diagram [25], and nonlinear dynamics of a dipolar [26] or spinor [27] BEC.

The theoretical model that gives a satisfactory description of the dynamics of BECs is the mean-field Gross–Pitaevskii (GP) equation [28]. Nonlinear terms arise in the GP equation to account for the effect of interatomic interactions in the condensate. One of the most attractive features of cold atomic gases is how the interatomic interaction affects the nonlinear dynamical properties of condensates. Two-body interatomic interactions in BECs are modeled through the s-wave scattering length, a_s, which may be either negative or positive, meaning that the interaction is attractive or repulsive, respectively [28]. The strength and sign of the atomic scattering length can be varied by tuning the external magnetic field near Feshbach resonance [29]. This indicates that the interaction strength can be controlled by using different experimental devices. As is well known, the repulsive BECs in OL can give rise to stable localized matter-wave states in the form of gap solitons. These BEC gap solitons were predicted theoretically [30–32] and demonstrated experimentally [6]. Gap solitons are represented by stationary solutions to the respective GP equation, with the eigenvalue (chemical potential) located in a finite bandgap of the OL-induced spectrum [33]. In the BECs with attractive interactions ($a_s < 0$), solitons have been realized in the ground state of the condensate. Such solitons were created in condensates of ^{7}Li [34] and ^{85}Rb [35] atoms, with the sign of the atomic interactions switched from negative to positive by means of the Feshbach-resonance technique (in the latter case, the solitons were observed in a post-collapse state of the condensate). In the presence of a periodic potential, such solitons should exist too, with the chemical potential falling in the semi-infinite gap of the spectrum, as first shown in the context of the optical setting [8], and later demonstrated in details in the framework of the GP equation [36,37].

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In the context of BECs in the presence of an OL (more specifically a sinusoidal external potential), it has been shown that some intriguing demonstrations such as the dephasing and localization phenomena occur because of the effect of modulational instability (MI). This instability refers to a general phenomenon in nonlinear wave equations and appears in many nonlinear dispersive systems. It indicates that, due to the interplay between nonlinearity and the dispersive effects, a small perturbation on the envelope of a plane wave may induce an exponential growth of the wave amplitude, resulting in a break-up of the carrier wave into a train of localized waves [38]. Modulational instability has been studied both experimentally [39–41] and theoretically [42–47]. Recently, many investigations have been devoted to the MI of both single-component BECs and double-species BECs in OLs [48–53]. Moreover, numerous studies with relation to the MI have also attracted much interest, as the MI is an indispensable mechanism for understanding the relevant dynamic processes in BEC systems, which include domain formation [54,55], generation and propagation of solitonic waves [56,57] and quantum phase transition [58], etc. In the specific case of condensates trapped in a periodic potential and driven with a harmonic magnetic field, the MI has been observed experimentally in Ref. [41], following the theoretical analysis presented in Ref. [42]. However, only few efforts have been devoted to the *investigation of parameter domains of MI* of Bose–Einstein condensates with optical and harmonic potentials.

In this Letter, we reexamine the modulational instability in the GP equation with the effects of periodic potential and harmonic magnetic field. We intend to find the *explicit criteria* for MI as well as MI domains of such a system, by means of the time-dependent variational approach (TDVA) and numerical calculations, following the idea of Ref. [59]. *For this aim, considering the case of weak harmonic potential, we perform the TDVA* to propose not only the MI conditions but also the time-dependence of the perturbation parameters. The work is structured as follows: in Section 2, we present the theoretical model that describes the condensates understudy. Section 3 is devoted to the mathematical framework in which we derive the MI conditions of the system through the variational approach. Then, in Section 4, we perform direct numerical integrations to check the validity of the MI conditions found by analytical methods. Finally, in Section 5, we summarize our results and present our conclusions.

2. Theoretical model

We consider a BEC immersed in an optical lattice and a highly elongated harmonic trap. At very low temperatures, the system can be described by the following nonlinear mean-field GP equation [13,50]

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + \frac{m}{2}\left(\omega_\perp^2\rho^2 + \omega_x^2x^2\right)\psi(\mathbf{r},t) + V_{\mathrm{opt}}(x)\psi(\mathbf{r},t) + g_0\left|\psi(\mathbf{r},t)\right|^2\psi(\mathbf{r},t), \tag{1}$$

where $\hbar$ is the reduced Planck's constant, m is the mass of the boson, $\omega_\perp$ and ω_x, respectively, are the radial and longitudinal frequencies of the anisotropic trap ($\omega_\perp \gg \omega_x$), and ρ denotes the radial distance. The parameter g_0 is the strength of the two-body interatomic interactions, which is related to the s-wave scattering length a_s by $g_0 = 4\pi\hbar^2 a_s/m$. The OL potential is applied only in the longitudinal direction, i.e., $V_{\mathrm{opt}}(x) = V_{\max}\cos^2(kx+\theta)$, with $V_{\max}$ being the effective depth of the optical potential. The parameter θ is an arbitrary phase and $k = 2\pi/\lambda$ is the wave number of the OL that can be controlled by varying the angle ϑ between the two counterpropagating laser beams whose interference creates the OL [33,60]. The wavelength of the interference pattern is expressed in terms of the angle between the laser beams by the relation $\lambda = (\lambda_{\mathrm{laser}}/2)\sin(\vartheta/2)$, where λ_{laser} is the wavelength of the laser beams producing the OL.

We take $\theta = 0$ and $\psi(\mathbf{r},t) = \phi_0(\rho)\phi(x,t)$ where $\phi_0 = \sqrt{\frac{1}{\pi a_\perp^2}} \times \exp(-\frac{\rho^2}{2a_\perp^2})$, with $\rho = \sqrt{z^2+y^2}$ and $a_\perp = \sqrt{\hbar/m\omega_\perp}$, is the ground state of the radial equation

$$-\frac{\hbar^2}{2m}\nabla_\rho^2\phi_0 + \frac{m}{2}\omega_\perp^2\rho^2\phi_0 = \hbar\omega_\perp\phi_0. \tag{2}$$

Multiplying both sides of the GP equation (1) by ϕ_0^* and integrating over the transverse variable ρ, we come to the following quasi-1D GP equation:

$$i\hbar\frac{\partial\phi(x,t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m}{2}\omega_x^2x^2 + V_{\max}\cos^2(kx)\right)\phi(x,t) + \frac{g_0}{2\pi a_\perp^2}\left|\phi(x,t)\right|^2\phi(x,t). \tag{3}$$

It is more convenient to use the above Eq. (3) into a dimensionless form [61–63]. For this purpose we make the transformation of variables as $T = t\nu$, $X = xk$, $\phi \sim \phi\sqrt{2a_{s0}\omega_\perp/\nu}$, where $\nu = E_R/\hbar$, $\alpha = \omega_x^2/4\nu^2$ and $E_R = \hbar^2k^2/2m$, then, we can get the following normalized 1D GP equation with harmonic and optical potentials:

$$i\frac{\partial\phi(X,T)}{\partial T} = \left(-\frac{\partial^2}{\partial X^2} + \alpha X^2 + V_s\cos^2(X)\right)\phi(X,T) + s\left|\phi(X,T)\right|^2\phi(X,T), \tag{4}$$

where $V_s = V_{\max}/E_R$, $a_s = sa_{s0}$, with a_{s0} the constant scattering length, $s = \pm 1$ is the sign of the scattering length. *In the case of deep optical potential wells combined to a weak confinement, the harmonic trapping potential can be negligible compared to the OL potential. Then, using for simplicity the initial notation* (x,t) *instead of* (X,T)*, we may rewrite the above equation to get*:

$$i\frac{\partial\phi(x,t)}{\partial t} = \left(-\frac{\partial^2}{\partial x^2} + V_s\cos^2(x) + s\left|\phi(x,t)\right|^2\right)\phi(x,t). \tag{5}$$

In its present form, this equation can easily be cast into a variational problem.

3. Variational approximation

The variational approximation method is an important tool for solving and describing nonlinear systems [64–66]. *Very recently, a variational method based on a Gaussian ansatz has been used to investigate geometric resonances in Bose–Einstein condensates* [67]. In this work, we use the time-dependent variational approach to examine the MI of the condensates trapped in a periodic potential. The use of the TDVA for the study of solitons is not novel (see for example [59,68] and references therein). Unlike the other variational methods, the TDVA uses the MI-motivated ansatz directly in the Lagrangian, and allows us to obtain the criteria of MI excitation. Let us note in passing that the classical linear stability calculation may fail in this problem since there is still a space-dependent coefficient in Eq. (5). Through the TDVA, we attempt to identify the intervals of unstable wave numbers. The process should start by finding the Lagrangian density generating Eq. (5), which is

$$\mathcal{L} = \frac{i}{2}\left(\frac{\partial\phi}{\partial t}\phi^* - \frac{\partial\phi^*}{\partial t}\phi\right) - \left|\frac{\partial\phi}{\partial x}\right|^2 - V_s\cos^2(x)|\phi|^2 - \frac{1}{2}s|\phi|^4. \tag{6}$$

Our variational ansatz is a MI-motivated wave function, due to the modulation of a plane wave profile defined by