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New soliton-like solutions for KdV equation with variable coefficient

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Abstract

In this Letter, based on the idea of [Y.T. Gao, B. Tian, Comput. Phys. Commun. 133 (2001) 158], Jacobi elliptic function expansion method is improved and, with symbolic computation, some new soliton-like solutions are obtained for KdV equation with variable coefficient. Our method can also be used to solve other nonlinear partial differential equations.

Keywords: Soliton-like solution; KdV equation; Solitary wave solution

In recent years, more and more attention has been paid to the searching for solitonic solutions of nonlinear equations of mathematical physics [1–7] that describe some important physical and dynamic processes. For example, a generalized hyperbolic function method in [7], which is more powerful than the typical tanh function method [16], is proposed by Gao and Tian and applied to discussed the cylindrical KdV equation to answer a couple of problems in the plasma-physics in [13,14] by Tian and Gao. In the present Letter, based on the work of Ref. [7], we improve the Jacobi elliptic function expansion method proposed recently and consider the generalized KdV equation with variable coefficient [10]

$$u_t + 2\beta(t)u + \left[\alpha(t) + \beta(t)x\right]u_x - 3A\gamma(t)uu_x + \gamma(t)u_{xxx} = 0. \tag{1}$$

The importance of Eq. (1) is well known and there is a continuing high level of interest in the construction of its solitary wave solutions. When $\gamma(t) = -K_0(t)$, $A = -2$, $\beta(t) = h(t)$, $\alpha(t) = -4K_1(t)$, Eq. (1) can be degenerated to the following variable coefficient and

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nonisospectral KdV equation studied in [8] and [9]:

$$u_t = K_0(t)(u_{xxx} + 6uu_x) + 4K_1(t)u_x - h(t)(2u + xu_x), \tag{2}$$

and when $\gamma(t) = 1$, $A = -2$, $\alpha(t) = c_0$, Eq. (1) can also be degenerated to the following variable coefficient KdV equation discussed in [11] and [12]:

$$u_{xxx} + 6uu_x + \left[\left(c_0 + \beta(t)x\right)u\right]_x + \beta(t)u + u_t = 0. \tag{3}$$

Considering homogeneous balance between uu_x and u_{xxx} in Eq. (1), we suppose that the solution of Eq. (1) is of the form

$$u = f + g_1\varphi(\xi) + g_2\varphi^2(\xi), \tag{4}$$

$$\xi = px + q, \tag{5}$$

$$\left(\frac{d\varphi}{d\xi}\right)^2 = a\varphi^4 + b\varphi^3 + c\varphi^2 + d, \tag{6}$$

where a, b, c and d are constants, and $f = f(t)$, $g_1 = g_1(t)$, $g_2 = g_2(t)$, $p = p(t)$ and $q = q(t)$ are functions of t to be determined. Substituting (4)–(6) into (1) and collecting coefficients of $\varphi^k\varphi^{(l)}$, $k = 0, 1, 2, 3$, $l = 0, 1$, then setting each coefficients to zero, we have the following set of over-determined equations for f, g_1, g_2, p, q, a, b, c and d:

$$2\beta f + f' = 0, \tag{7}$$

$$2\beta g_1 + g_1' = 0, \tag{8}$$

$$2\beta g_2 + g_2' = 0, \tag{9}$$

$$g_1\left[c\gamma p^3 - 3A\gamma f p + (\alpha + \beta x)p + (p'x + q')\right] = 0, \tag{10}$$

$$3\gamma g_1 p^3 b + 8\gamma g_2 p^3 c - 3A\gamma g_1^2 p - 6A\gamma f g_2 p + 2(\alpha + \beta x)g_2 p + 2g_2(p'x + q') = 0, \tag{11}$$

$$2\gamma g_1 p^3 a + 5\gamma g_2 p^3 b - 3A\gamma g_1 g_2 p = 0, \tag{12}$$

$$4\gamma g_2 p^3 a - A\gamma g_2^2 p = 0. \tag{13}$$

With symbolic computation [7,13–15], we obtain the following two cases.

Case 1.

$$f = k_1 e^{-2\int\beta\,dt}, \qquad g_1 = k_2 e^{-2\int\beta\,dt},$$
$$g_2 = k_3 e^{-2\int\beta\,dt}, \qquad p = k_4 e^{\int\beta\,dt}, \qquad a = \frac{Ak_3}{4k_4^2},$$
$$b = \frac{Ak_2}{2k_4^2}, \qquad c = \frac{Ak_2^2}{4k_3k_4^2},$$
$$q = \int \frac{1}{4k_3}\left(12A\gamma k_1k_3k_4 e^{-3\int\beta\,dt} - A\gamma k_2^2 k_4 e^{-3\int\beta\,dt} - 4\alpha k_3k_4 e^{-3\int\beta\,dt}\right)dt,$$

where k_1 and k_2 are arbitrary constants, $k_3 \neq 0$, $k_4 \neq 0$.

If $Ak_3 > 0$, $d = 0$, then [2]

$$\varphi(\xi) = \frac{cb\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}\xi\right)}{b^2 - ac\left(1 - \tanh\left(\frac{\sqrt{c}}{2}\xi\right)\right)^2} = \frac{2k_2\,\mathrm{sech}\left(\frac{|k_2|}{4|k_4|}\sqrt{\frac{A}{k_3}}\xi\right)}{4k_3 - k_3\left(1 - \tanh\left(\frac{|k_2|}{4|k_4|}\sqrt{\frac{A}{k_3}}\xi\right)\right)^2}. \tag{14}$$

From (4), (5), (14) and Case 1, we can obtain the following soliton-like solutions for Eq. (1)

$$u = k_1 e^{-2\int\beta\,dt} + k_2 e^{-2\int\beta\,dt}\left[\frac{2k_2\,\mathrm{sech}\left(\frac{|k_2|}{4|k_4|}\sqrt{\frac{A}{k_3}}\xi\right)}{4k_3 - k_3\left(1 - \tanh\left(\frac{|k_2|}{4|k_4|}\sqrt{\frac{A}{k_3}}\xi\right)\right)^2}\right] + k_3 e^{-2\int\beta\,dt}\left[\frac{2k_2\,\mathrm{sech}\left(\frac{|k_2|}{4|k_4|}\sqrt{\frac{A}{k_3}}\xi\right)}{4k_3 - k_3\left(1 - \tanh\left(\frac{|k_2|}{4|k_4|}\sqrt{\frac{A}{k_3}}\xi\right)\right)^2}\right]^2,$$

$$\xi = k_4 e^{-\int\beta\,dt}x + \int \frac{1}{4k_3}\left(12A\gamma k_1k_3k_4 e^{-3\int\beta\,dt} - A\gamma k_2^2 k_4 e^{-3\int\beta\,dt} - 4\alpha k_3k_4 e^{-\int\beta\,dt}\right)dt,$$

where k_1 and k_2 are arbitrary constants, $k_3 \neq 0$, $k_4 \neq 0$.

Case 2.

$$f = k_1 e^{-2\int\beta\,dt}, \qquad g_1 = 0, \qquad g_2 = k_3 e^{-2\int\beta\,dt},$$
$$p = k_4 e^{-\int\beta\,dt}, \qquad a = \frac{Ak_3}{4k_4^2}, \qquad b = 0,$$
$$q = \int\left(3A\gamma k_1 k_4 e^{-3\int\beta\,dt} - \alpha k_4 e^{-\int\beta\,dt} - 4\gamma k_4^3 c e^{-3\int\beta\,dt}\right)dt,$$

where k_1, k_3 and c are arbitrary constants, $k_4 \neq 0$.

If $c > 0$, $Ak_3 < 0$, $d = 0$, then [2]

$$\varphi(\xi) = 2|k_4|\sqrt{-\frac{c}{Ak_3}}\,\mathrm{sech}\left(\sqrt{c}\,\xi\right). \tag{15}$$