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Convergence of intuitionistic fuzzy sets

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ABSTRACT

We define pointwise convergence, uniform convergence, Γ-convergence and convergence in supremum metric for the intuitionistic fuzzy sets. The uniform convergence is in the topology induced by lower and upper pseudo metrics. The Γ-convergence is the Kuratowski–Painlevé convergence of the endographs of the intuitionistic fuzzy sets. The supremum metric is the supremum of Hausdroff distance among the ζ-cuts of the intuitionistic fuzzy sets. We discuss the mutual relationship of these convergences. Topological structures are also discussed in detail. Adequate number of examples are given to illustrate the relationship among these convergences.

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1. Introduction

Fuzzy mathematics has proved its usefulness over the years and able to solve many problems which classical logic unable to handle. This theory was started by Zadeh [28] in 1965. Fuzzy sets have found several flourishing applications in various fields including control [18], robot selection [23], intelligent systems [7], satellite image analysis [1] and the list goes on. A fuzzy set is a function from a topological space (X, τ) to [0, 1]. Convergence of fuzzy sets is an important subject in fuzzy set theory. There are several types of fuzzy sets convergences in literature [12–15,26]. Pedraza et al. discussed the relationship between pointwise convergence, Γ-convergence and convergence in supremum metric of fuzzy sets in [22]. In 1986, Atanassov [3] introduced the concept of Intuitionistic Fuzzy Sets (IFS) as a generalization of fuzzy sets. That deals with more ambiguous situations. IFS is a function f from a topological space (X, τ) to $\mathbb{T}$, where $\mathbb{T} = \{(\alpha, \beta) \in [0, 1]^2 : \alpha + \beta \leq 1\}$. We refer to the book of Atanassov [4] for the basics on IFS.

Recently, statistical convergence of intuitionistic fuzzy sets in the settings of norm space is discussed in [2,8,19–21,25]. Intuitionistic fuzzy norms deal the situations where norms of the vectors cannot be found exactly. The notion of statistical convergence is useful to measure the numerical convergence by means of the density. Our approach is more classical in the sense that we use topological structures without imposing the conditions of norm space.

The main purpose of this paper is to study pointwise convergence, uniform convergence, Γ-convergence and convergence in supremum metric for the intuitionistic fuzzy sets and discuss the mutual relationships among these convergences. This paper is organized as follows. In Section 2, we start with preliminary and auxiliary results needed in the rest of the paper. In Section 3, topological structure on $\mathbb{T}$ is discussed in detail. In Section 4, relationship between Γ-convergence and pointwise convergence of IFS is studied. Section 5 is dedicated to convergence in supremum metric and its relation with other convergences. Some concluding remarks are given in the last section.

2. Preliminaries

Deschrijver and Kerre [9] have shown that intuitionistic fuzzy sets can also be consider as L-fuzzy sets in the sense

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of Goguen [11]. They also defined the order as follows: for $\zeta = (\alpha_\zeta, \beta_\zeta)$, $\eta = (\alpha_\eta, \beta_\eta) \in \mathbb{T}$, $\zeta \leq \eta$ if and only if $\alpha_\zeta \leq \alpha_\eta$ and $\beta_\zeta \geq \beta_\eta$. With this ordering $\mathbb{T}$ is a complete lattice, but it is not a totally ordered set. The maximum and minimum is $\max(\zeta, \eta) = (\max(\alpha_\zeta, \alpha_\eta), \min(\beta_\zeta, \beta_\eta))$ and $\min(\zeta, \eta) = (\min(\alpha_\zeta, \alpha_\eta), \max(\beta_\zeta, \beta_\eta))$, respectively. For a subset A of $\mathbb{T}$, $\sup A := (\max \alpha_x, \min \beta_x)$ and $\inf A := (\min \alpha_x, \max \beta_x)$ for all $x \in A$. Note that (1, 0) is the largest and (0, 1) is the smallest element in $\mathbb{T}$.

In this paper, d_e is used for usual Euclidean metric on $\mathbb{R}^2$ or any of its subsets. For a metric space (X, d) we denote $B_d(x, r)$ as an open ball for radius $r > 0$ and for $A \subseteq X$ define $B_d(A, r) = \cup_{x\in A} B_d(x, r)$. In case of topological space (X, τ), $\mathcal{N}_x$ is the set of all neighborhoods of $x \in X$.

Let f be an IFS then the *endograph* or *hypograph* of f is defined as $\text{end}\, f := \{(x, \zeta) \in X \times \mathbb{T} : \zeta \leq f(x)\}$, similarly *epigraph* is defined as $\text{epi}\, f := \{(x, \zeta) \in X \times \mathbb{T} : f(x) \leq \zeta\}$ and the $\zeta-$*cut* is defined as $[f]^\zeta := \{x \in X \colon \zeta \leq f(x)\}$. A function can described uniquely by its endograph or $\zeta-$cuts. We will consider the endographs in the topological space $(X \times \mathbb{T}, \tau \times \tau_{d_e})$.

Consider a net $\{A_\lambda\}_{\lambda \in \Lambda}$ in topological space (X, τ) then

- A subset of Λ is said to be *residual* if it contains all indices at or beyond some index λ.
- A subset of Λ is said to be *cofinal* if it contains some indices at or beyond each index λ.
- the *lower limit* of $\{A_\lambda\}_{\lambda \in \Lambda}$ is the set

  $$\text{Li}\, A_\lambda = \{x \in X : U \cap A_\lambda \neq \emptyset \text{ residually}, \forall\, U \in \mathcal{N}_x\};$$
- the *upper limit* of $\{A_\lambda\}_{\lambda \in \Lambda}$ is the set

  $$\text{Ls}\, A_\lambda = \{x \in X : U \cap A_\lambda \neq \emptyset \text{ cofinally}, \forall\, U \in \mathcal{N}_x\}.$$

We say a net $\{A_\lambda\}_{\lambda \in \Lambda}$ in (X, τ) is lower (resp. upper) Kuratowski–Painlevé convergent to $A \subseteq X$ if $A \subseteq \text{Li}\, A_\lambda$ (resp. $\text{Ls}\, A_\lambda \subseteq A$). For more details see [5, page 2, 145]. Γ-convergence is the Kuratowski–Painlevé convergence of the endographs of IFS. Γ-convergence and its counter part epi-convergence have their roots in Convex Analysis [5,6,10,17]. These convergences are due to infimal convergence for convex functions, introduced by Wijsman [27] . Maximizers are important in Variational Calculus (see [24]) and also in discussion of the relationship between Γ-convergence and convergence in supremum metric.

Consider a metric space (X, d), we define the supremum metric between the two IFS as the supremum of the distance between their $\zeta-$cuts in Hausdroff extended pseudometric H_d. *Hausdroff extended pseudometric* between two subsets of X is defined as

$$H_d(A, B) = \max(e_d(A, B), e_d(B, A))$$

where

$$e_d(A, B) = \begin{cases} \sup_{a\in A} d(a, B) & \text{if } A \neq \emptyset \\ 0, & \text{if } A = \emptyset \end{cases}$$

is the excess of A over B.

$H_d(A, B)$ can be characterize as

$$H_d(A, B) = \max\{\inf\{\epsilon > 0 : A \subseteq B_d(B, \epsilon)\}, \inf\{\epsilon > 0 : B \subseteq B_d(A, \epsilon)\}\} \quad (1)$$

where inf is $+\infty$ if no such ϵ exists.

3. Topological structure and convergence on $\mathbb{T}$

In this section, basic definitions of convergences and structures on $\mathbb{T}$ are given, which will be useful in our study.

Definition 3.1. For a net $\{x_\lambda\}_{\lambda \in \Lambda}$ in $\mathbb{T}$, we define

- the lower limit of $\{x_\lambda\}_{\lambda \in \Lambda}$ as

  $$\liminf_{\lambda\in\Lambda} x_\lambda = \sup_{\lambda'\in\Lambda} \inf_{\lambda\geq\lambda'} x_\lambda;$$
- the upper limit of $\{x_\lambda\}_{\lambda \in \Lambda}$ as

  $$\limsup_{\lambda\in\Lambda} x_\lambda = \inf_{\lambda'\in\Lambda} \sup_{\lambda\geq\lambda'} x_\lambda.$$

Definition 3.2. A net $\{x_\lambda\}_{\lambda \in \Lambda}$ in $\mathbb{T}$ is:

- Lower convergent to $x \in \mathbb{T}$ if $\liminf_{\lambda\in\Lambda} x_\lambda \geq x$, C_{lower} is the collection of all such x;
- Upper convergent to $x \in \mathbb{T}$ if $\limsup_{\lambda\in\Lambda} x_\lambda \leq x$, C_{upper} is the collection of all such x;
- Convergent to $x \in \mathbb{T}$ if $\liminf_{\lambda\in\Lambda} x_\lambda = \limsup_{\lambda\in\Lambda} x_\lambda = x$.

This order $\leq$ is quite useful because in this ordering the lower and the upper convergences are topological. Consider the lower topology induces by *lower pseudometric* $d_{lower}(\zeta, \eta) = \max(\alpha_\zeta - \alpha_\eta, \beta_\eta - \beta_\zeta, 0)$, then it is easy to verify that convergence in this topology and lower convergence coincides. Similarly, for the upper convergence we have upper topology with the *upper pseudometric* $d_{upper}(\zeta, \eta) = \max(\alpha_\eta - \alpha_\zeta, \beta_\zeta - \beta_\eta, 0)$. The types of open sets and regions of convergence are shown in Fig. 1.

Definition 3.3. For a given $x \in \mathbb{T}$, a net $\{x_\lambda\}_{\lambda \in \Lambda}$ in $\mathbb{T}$ is said to be:

- $\varrho_\leq$-convergent to x if there exits λ_0 such that $x \leq x_\lambda$ for all $\lambda \geq \lambda_0$;
- $\varrho_\geq$-convergent to x if there exits λ_0 such that $x \geq x_\lambda$ for all $\lambda \geq \lambda_0$.

Note that $\varrho_\leq$-convergence is given by Alexandroff topology associated with partial order $\leq$ of $\mathbb{T}$, whose open sets are

$$\tau_\leq = \{O \subseteq \mathbb{T} : \text{ if } \zeta, \eta \in \mathbb{T}, \zeta \in O \text{ and } \zeta \leq \eta \text{ then } \eta \in O\}.$$

Similarly, $\varrho_\geq$-convergence is given by Alexandroff topology associated with partial order $\geq$ of $\mathbb{T}$, whose open sets are

$$\tau_\geq = \{O \subseteq \mathbb{T} : \text{ if } \zeta, \eta \in \mathbb{T}, \zeta \in O \text{ and } \zeta \geq \eta \text{ then } \eta \in O\}.$$

The structure of open sets in $\tau_\leq$ and $\tau_\geq$ are shown in Fig. 2.

Convergence in $\tau_\leq$ (resp. $\tau_\geq$) implies lower (resp. upper) convergence but the converse is not true in general. We give two more definition of convergences, which are also useful in our study.

Definition 3.4. For a given $x \in \mathbb{T}$, we say that a net $\{x_\lambda\}_{\lambda \in \Lambda}$ in $\mathbb{T}$ is:

- $\varrho_<$-convergent to x if there exits λ_0 such that $x < x_\lambda$ for all $\lambda \geq \lambda_0$;
- $\varrho_>$ *-convergent* to x if there exits λ_0 such that $x > x_\lambda$ for all $\lambda \geq \lambda_0$.

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