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Electromagnetic waves in non-integer dimensional spaces and fractals

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ABSTRACT

Electromagnetic waves in non-integer dimensional spaces are considered in the framework of continuous models of fractal media and fields. Using the recently suggested vector calculus for non-integer dimensional space, we consider electromagnetic fields in isotropic case. This D-dimensional calculus allows us to describe fractal properties by continuous models with non-integer dimensional spaces. We prove that the wave equation for non-integer dimensional space is similar to equation of waves in non-fractal medium with heterogeneity of power-law type. The speed of electromagnetic waves and the effective refractive index of non-integer dimensional spaces and fractals are discussed.

1. Introduction

Fractals are measurable metric sets with non-integer dimensions [1,2]. We can describe fractal media by using methods of "analysis on fractals" [3,4]. At present an application of the "analysis on fractals" to solve differential equations on fractals [4] for real physical problems is limited by a weak development of this area of mathematics. We can consider fractal media as continuous media in non-integer dimensional space. The non-integer dimension does not reflect all properties of the fractal media, but it is a main characteristic of fractal media. For this reason, continuous models with non-integer dimensional spaces can allow us to get some important conclusions about the behavior of the fractal media.

Continuous models for fractal distributions of charges, currents, media and fields have been proposed in [5–9]. These models are based on the notion of power-law density of states [10]. To take into account this density of states, we use the fractional-order integrals that is connected with fractional-dimensional integration [8,10]. It should be noted that fractional-order integrals and derivatives are used to describe fractional nonlocal models, which are based on fractional-order vector calculus [11] in general. The suggested continuous models of fractal media and electromagnetic fields have been developed in works [12–14] and [15–24] to describe anisotropic fractal media and electromagnetic waves in fractional space. Continuous models that are used in [15–24], are based on fractional dimensional generalizations of the scalar Laplace operators, which are proposed in papers [25,26]. It should be noted that the first-order differential vector operators (gradient, divergence, curl), and the vector Laplacian are not considered in [25,26]. This greatly restricts us in application of non-integer dimensional space approach to describe fractal media and fields. For example, the scalar Laplacian cannot be used for the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic fields $\mathbf{B}(\mathbf{r}, t)$ in the framework of continuous models with non-integer dimensional spaces.

An attempt to suggest first-order differential vector operators for non-integer dimensional spaces has been proposed in [18–24]. In these works, the operators are suggested

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only as approximations of the square of the Laplace operator. Recently a generalization of differential vector operators of first orders (grad, div, curl), the scalar and vector Laplace operators for non-integer dimension spaces have been suggested in papers [28–30] without any approximation. This allows us to extend the application area of continuous models with non-integer dimensional spaces. Using this new D-dimensional vector calculus, we can describe isotropic and anisotropic fractal media by using the non-integer dimensional space approach.

In this paper, we use the non-integer dimensional vector calculus, which is proposed in paper [28], to describe electromagnetic waves in non-integer dimensional spaces, fractals and isotropic fractal media. We prove that the wave equations for non-integer dimensional spaces are similar to the equations of waves in usual (non-fractal) media with power-law heterogeneity.

2. Vector differentiation for non-integer dimensional space

In the continuous models of fractal media, it is convenient to work with the physically dimensionless variables $x/R_0 \to x$, $y/R_0 \to x$, $z/R_0 \to x$, $\mathbf{r}/R_0 \to \mathbf{r}$, where R_0 is a characteristic size of considered model. This yields dimensionless integration and dimensionless differentiation in D-dimensional space. In this case the physical quantities of fractal media have correct physical dimensions.

Let us give some introduction to non-integer dimensional differentiation of integer orders (for details, see [25–29]). The vector differential operators for non-integer dimension have been derived in [28] by analytic continuation in dimension from integer n to non-integer D.

For simplification we will consider spherically symmetric case of fractal media, where scalar field φ and vector fields $\mathbf{E}$, $\mathbf{B}$ are independent of angles

$$\varphi(\mathbf{r},t)=\varphi(r,t), \quad \mathbf{E}(\mathbf{r},t)=E_r(r,t)\,\mathbf{e}_r, \quad \mathbf{B}(\mathbf{r},t)=B_r(r,t)\,\mathbf{e}_r,$$

where $\mathbf{e}_r = \mathbf{r}/r$, $r = |\mathbf{r}|$. Here $E_r = E_r(r)$ and $B_r = B_r(r)$ are the radial component of $\mathbf{E}$ and $\mathbf{B}$. In this case, we will work with rotationally covariant functions only. This simplification is analogous to the simplification of integration over non-integer dimensional space suggested in [27]. One of the main our simplification is that the electromagnetic components are radial functions. We note that for random fractals, this assumption is natural [38,39].

In general, the dimension D of the region V_D of fractal media and the dimension d of boundary $S_d = \partial V_D$ of this region are not related by the equation $d = D - 1$, i.e.,

$$\dim(\partial V_D) \neq \dim(V_D) - 1, \tag{1}$$

where $\dim(V_D) = D$ and $\dim(\partial V_D) = d$. We will use the parameter

$$\alpha_r = D - d, \tag{2}$$

which is a dimension of fractal medium along the radial direction.

In [28], the differential operators for non-integer D have been proposed in the following forms.

For non-integer dimensional space, the divergence operator for the vector field $\mathbf{E} = \mathbf{E}(r)$ can be represented [28] in the form

$$\mathrm{Div}_r^{D,d}\,\mathbf{E} = \pi^{(1-\alpha_r)/2}\,\frac{\Gamma((d+\alpha_r)/2)}{\Gamma((d+1)/2)} \times \left(\frac{1}{r^{\alpha_r-1}}\frac{\partial E_r(r)}{\partial r} + \frac{d}{r^{\alpha_r}}\,E_r(r)\right). \tag{3}$$

This is (D, d)-dimensional divergence operator for fractal media with $d \neq D-1$. For $\alpha_r = 1$, i.e. $d = D-1$, Eq. (3) gives

$$\mathrm{Div}_r^{D}\,\mathbf{E} = \frac{\partial E_r(r)}{\partial r} + \frac{D-1}{r}\,E_r(r). \tag{4}$$

The gradient for the scalar field $\varphi(\mathbf{r}) = \varphi(r)$ depends on the radial dimension α_r [28] in the form

$$\mathrm{Grad}_r^{D,d}\,\varphi = \frac{\Gamma(\alpha_r/2)}{\pi^{\alpha_r/2}\,r^{\alpha_r-1}}\,\frac{\partial\varphi(r)}{\partial r}\,\mathbf{e}_r. \tag{5}$$

For $\alpha_r = 1$, i.e. $d = D-1$, the gradient in non-integer dimensional space is

$$\mathrm{Grad}_r^{D}\,\varphi = \frac{\partial\varphi(r)}{\partial r}\,\mathbf{e}_r. \tag{6}$$

The curl operator for the vector field $\mathbf{E} = \mathbf{E}(r)$ is equal to zero, $\mathrm{Curl}_r^D\,\mathbf{E} = 0$.

Using the operators (3) and (5) for the fields $\varphi = \varphi(r)$ and $\mathbf{E} = E(r)\,\mathbf{e}_r$, in paper [28] we obtain the scalar and vector Laplace operators for the case $d \neq D-1$ by the equation

$${}^S\Delta_r^{D,d}\varphi = \mathrm{Div}_r^{D,d}\,\mathrm{Grad}_r^{D,d}\,\varphi, \quad {}^V\Delta_r^{D,d}\mathbf{E} = \mathrm{Grad}_r^{D,d}\,\mathrm{Div}_r^{D,d}\,\mathbf{E}. \tag{7}$$

Then the scalar Laplacian for $d \neq D-1$ for the field $\varphi = \varphi(r)$ is

$${}^S\Delta_r^{D,d}\varphi = \frac{\Gamma((d+\alpha_r)/2)\,\Gamma(\alpha_r/2)}{\pi^{\alpha_r-1/2}\,\Gamma((d+1)/2)} \times\left(\frac{1}{r^{2\alpha_r-2}}\,\frac{\partial^2\varphi}{\partial r^2} + \frac{d+1-\alpha_r}{r^{2\alpha_r-1}}\,\frac{\partial\varphi}{\partial r}\right), \tag{8}$$

For $\alpha_r = 1$, i.e. $d = D-1$, Eq. (8) gives

$$\Delta_r^D\varphi = \mathrm{Div}_r^D\mathrm{Grad}_r^D\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{D-1}{r}\,\frac{\partial\varphi}{\partial r}S, \tag{9}$$

where we use $\Gamma(1/2) = \sqrt{\pi}$.

The vector Laplacian in non-integer dimensional space with $d \neq D-1$ and the field $\mathbf{E} = E_r(r)\,\mathbf{e}_r$ is

$${}^V\Delta_r^{D,d}\mathbf{E} = \frac{\Gamma((d+\alpha_r)/2)\,\Gamma(\alpha_r/2)}{\pi^{\alpha_r-1/2}\,\Gamma((d+1)/2)}\left(\frac{1}{r^{2\alpha_r-2}}\,\frac{\partial^2 E_r(r)}{\partial r^2} + \frac{d+1-\alpha_r}{r^{2\alpha_r-1}}\,\frac{\partial E_r(r)}{\partial r} - \frac{d\alpha_r}{r^{2\alpha_r}}\,E_r(r)\right)\mathbf{e}_r. \tag{10}$$

For $\alpha_r = 1$, i.e. $d = D-1$, Eq. (10) gives

$${}^V\Delta_r^{D}\mathbf{E} = \mathrm{Grad}_r^D\,\mathrm{Div}_r^D\,\mathbf{E} = \left(\frac{\partial^2 E_r(r)}{\partial r^2} + \frac{D-1}{r}\,\frac{\partial E_r(r)}{\partial r} - \frac{D-1}{r^2}\,E_r(r)\right)\mathbf{e}_r. \tag{11}$$

For $D = 3$ Eqs. (4)–(11) give the well-known expressions for the gradient, divergence, scalar Laplacian and vector Laplacian in $\mathbb{R}^3$ for fields $\varphi = \varphi(r)$ and $\mathbf{E}(r) = E_r(r)\,\mathbf{e}_r$.

The vector differential operators (3), (5), (8) and (10), which are suggested in [28], allow us to describe complex fractal media with the boundary dimension $d \neq D-1$ by the non-integer dimensional space approach.