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Exact solutions for the quadratic mixed-parity Helmholtz–Duffing oscillator by bifurcation theory of dynamical systems

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ABSTRACT

The dynamical behavior and exact solutions of the quadratic mixed-parity Helmholtz–Duffing oscillator are studied by using bifurcation theory of dynamical systems. As a result, all possible phase portraits in the parametric space are obtained. All possible explicit parametric representations of the bounded solutions (soliton solutions, kink and anti-kink solutions and periodic solutions) are given. When parameters are varied, under different parametric conditions, various sufficient conditions guarantee the existence of the above solutions are given.

1. Introduction

The nonlinear Helmholtz–Duffing oscillator has received lots of attention especially in the last decade. The interest arises from large number of applications in the mathematical interpretation of the engineering problems such as ship dynamics, oscillation of the human eardrum, oscillations of one dimensional structural system with an initial curvature, some electrical circuits, microperforated panel absorber and heavy symmetric gyroscope [1–4].

Recently, Alías-Zúñiga [5] considered the quadratic mixed-parity Helmholtz–Duffing oscillator as follows:

$$\frac{d^2x}{dt^2} + f(x) = 0, \; f(x) = Ax + Bx^2 + \varepsilon x^3 + D_1, \tag{1.1}$$

where x denotes the displacement of the system, A is the natural frequency, ε is a non-linear system parameter, and B and D_1 are system parameters independent of time. By using Jacobi elliptic functions, Alías-Zúñiga derived the exact periodic solution of Eq. (1.1). Obviously, When $B = 0$ and $D_1 = 0$, Eq. (1.1) is a Duffing oscillator, while Eq. (1.1) becomes a Helmholtz oscillator with a single-well potential when $\varepsilon = 0$ and $D_1 = 0$. When $A = 1$, $B = 1 - \varepsilon$ and $D_1 = 0$, Eq. (1.1) becomes a Helmholtz–Duffing oscillator which has been studied by many other authors [6–11].

However, to the best of author's knowledge, the literature dealing with Eq. (1.1) is very limited. In this paper, we shall investigate the exact solutions of Eq. (1.1) in detail by using the bifurcation theory of dynamical systems [12–17].

Eq. (1.1) is equivalent to the planar system

$$\frac{dx}{dt} = y, \; \frac{dy}{dt} = -Ax - Bx^2 - \varepsilon x^3 - D_1, \tag{1.2}$$

with the first integral

$$H(x, y) = \frac{1}{2}y^2 + \frac{A}{2}x^2 + \frac{B}{3}x^3 + \frac{\varepsilon}{4}x^4 + D_1 x = h. \tag{1.3}$$

System (1.2) is a four-parameter planar dynamical system depending on the parameter set (A, B, ε, D_1). Since the phase orbits defined by the vector fields of (1.2) determine all the solutions of (1.1), we should investigate the bifurcations of phase portraits of (1.2) in the (x, y)−phase plane as the parameters are changed. Here we consider a physical model where only bounded solutions are meaningful. So we only

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pay attention to the bounded solutions of (1.2) and restrict our analysis to the case $\varepsilon \neq 0$.

Suppose that $x(t)$ is a continuous solution of (1.2) for $t \in \mathbf{R}$ and $\lim_{t\to\pm\infty} x(t) = a_{\pm}$. It is well-known that (i) $x(t)$ is called a soliton solution if $a_+ = a_-$; (ii) $x(t)$ is called a kink (or anti-kink) solution if $a_+ \neq a_-$. Usually, a homoclinic, heteroclinic and periodic orbit of (1.2) respectively corresponds to a soliton, kink and periodic solution of (1.1). Thus, to investigate all possible bifurcations of soliton, kink (or anti-kink) and periodic solutions of (1.1), we need to find all periodic annuli, homoclinic orbits and heteroclinic orbits of (1.2), which depend on the system parameters [18,19].

This paper is organized as follows. In Section 2, we discuss the bifurcations of phase portraits of (1.2). In Sections 3, by considering the dynamics of the solutions determined by the system, we shall give all possible exact explicit parametric representations of bounded solutions of (1.1) in the different parameter regions by using the elliptic functions and hyperbolic functions [20].

2. Phase portraits and bifurcation sets of Eq. (1.2)

In this section, we shall study all phase portraits and bifurcation sets of (1.2) in the parameter space.

Clearly, on the (x, y) phase plane, the abscissas of equilibrium points of the system (1.2) are the zeros of the function $f(x) = Ax + Bx^2 + \varepsilon x^3 + D_1$. For $\varepsilon \neq 0$ and $D_1 \neq 0$, denote that

$$\Delta_1 = 3A\varepsilon - B^2,\ \Delta_2 = 9A\varepsilon - 27D_1\varepsilon^2 - 2B^3$$
$$\Delta_3 = B^2A^2 + 18AB\varepsilon D_1 - 27D_1^2\varepsilon^2 - 4D_1B^3 - 4A^3\varepsilon.$$

The roots of the cubic equation $f(x) = 0$ can be expressed using Cardano's formulas [21], as

$$x_1 = -\frac{B}{3\varepsilon} + \frac{2^{2/3}}{6\varepsilon}(\Delta_2 + i\sqrt{\Delta_3})^{1/3} - \frac{2^{1/3}}{3\varepsilon}\frac{\Delta_1}{(\Delta_2 + i\sqrt{\Delta_3})^{1/3}},$$

$$x_2 = -\frac{B}{3\varepsilon} - \frac{4^{1/3}}{12\varepsilon}(1 + i\sqrt{3})(\Delta_2 + i\sqrt{\Delta_3})^{1/3} + \frac{1 - i\sqrt{3}}{3\cdot 4^{1/3}\varepsilon}\frac{\Delta_1}{(\Delta_2 + i\sqrt{\Delta_3})^{1/3}},$$

$$x_3 = -\frac{B}{3\varepsilon} - \frac{4^{1/3}}{12\varepsilon}(1 - i\sqrt{3})(\Delta_2 + i\sqrt{\Delta_3})^{1/3} + \frac{1 + i\sqrt{3}}{3\cdot 4^{1/3}\varepsilon}\frac{\Delta_1}{(\Delta_2 + i\sqrt{\Delta_3})^{1/3}}.$$

Thus, it is easily to see that the distribution of the equilibrium points of (1.2) as follows.

(1) When $\Delta_3 < 0$, (1.2) has only one equilibrium point $E_1(x_1, 0)$.
(2) When $\Delta_3 > 0$, (1.2) has three equilibrium points $E_1(x_1, 0)$, $E_2(x_2, 0)$ and $E_3(x_3, 0)$.
(3) When $\Delta_3 = 0$, (1.2) has two equilibrium points $E_1^*(x_1^*, 0)$ and $E_2^*(x_2^*, 0)$, where $x_1^* = -\Delta_2/(3\varepsilon\Delta_1)$ is a single real root of $f(x)$ and $x_2^* = \Delta_2/(6\varepsilon\Delta_1)$ is a multiple real root of $f(x)$.

For $\varepsilon \neq 0$ and $D_1 = 0$, denote that $\Delta_4 = B^2 - 4A\varepsilon$. Then, the distribution of the equilibrium points of (1.2) is as follows.

(1) When $\Delta_4 < 0$, (1.2) has only one equilibrium point $E_0(0, 0)$.
(2) When $\Delta_4 > 0$, (1.2) has three equilibrium points $E_0(0, 0)$, $E_{01}(x_{01}, 0)$ and $E_{02}(x_{02}, 0)$, where $x_{01} = (-B + \sqrt{\Delta_4})/(2\varepsilon)$ and $x_{02} = (-B - \sqrt{\Delta_4})/(2\varepsilon)$.
(3) When $\Delta_4 = 0$, (1.2) has two equilibrium points $E_0(0, 0)$ and $E_{01}^*(x_{01}^*, 0)$, where $x_{01}^* = -B/(2\varepsilon)$ is a multiple real root of $f(x)$.

Let $M(x_e, 0)$ be the coefficient matrix of the linearized system of (1.2) at an equilibrium point $(x_e, 0)$ and $J(x_e, 0)$ be its Jacobin determinant. By the theory of planar dynamical systems, we know that for an equilibrium point of a planar integrable system, if $J < 0$ then the equilibrium point is a saddle point; if $J > 0$ and $Trace(M(x_e, 0)) = 0$ then it is a center point; if $J = 0$ and the Poincare index of the equilibrium point is 0 then it is a cusp. And from Eq. (1.3), we have

$$h_i = H(x_i, 0)(i = 1, 2, 3),\ h_i^* = H(x_i^*, 0)(i = 1, 2),$$
$$h_0 = H(0, 0),\ h_{0i} = H(x_{0i}, 0)(i = 1, 2),\ h_{01}^* = H(x_{01}^*, 0).$$

By using the above fact, we now consider bifurcations of phase portraits of (1.2) for $\varepsilon > 0$ and $\varepsilon < 0$ respectively.

2.1. Phase portraits and bifurcation sets of Eq. (1.2) when $\varepsilon > 0$

2.1.1. Phase portraits and bifurcation sets of Eq. (1.2) when $\varepsilon > 0$ and $D_1 \neq 0$

When $\varepsilon > 0$ and $D_1 > 0$, the bifurcation curve $\Delta_3(B, A) = 0$ is made up of three parts:

$$L_{11}^{\pm} : A = f_1^{\pm}(B) = \frac{1}{12\varepsilon}\left(B^2 - \frac{B(B^3 + 216D_1\varepsilon^2)}{2(\delta_2 + 24\varepsilon\sqrt{-3D_1\delta_1})^{1/3}} \pm \frac{\sqrt{3}i}{2}(\delta_2 + 24\varepsilon\sqrt{-3D_1\delta_1})^{1/3} - \frac{B(B^3 + 216D_1\varepsilon^2)}{2(\delta_2 + 24\varepsilon\sqrt{-3D_1\delta_1})^{1/3}}\right)$$

and

$$L_{12} : A = f_2(B) = \frac{1}{12\varepsilon}\left(B^2 + (\delta_2 + 24\varepsilon\sqrt{-3D_1\delta_1})^{1/3} + \frac{B(B^3 + 216D_1\varepsilon^2)}{(\delta_2 + 24\varepsilon\sqrt{-3D_1\delta_1})^{1/3}}\right),$$

where $\delta_1 = B^9 - 81D_1\varepsilon^2B^6 + 2187D_1^2\varepsilon^4B^3 - 19683D_1^3\varepsilon^6$, $\delta_2 = B^6 - 540D_1\varepsilon^2B^3 - 5832D_1^2\varepsilon^4$, L_{11}^+ and L_{12} intersect at the point $B = 0$, L_{11}^- and L_{12} intersect at the point B which satisfies $\delta_1 = 0$. The bifurcation curve divides the (B, A)—parameter plane into the two subregions (see Fig. 1):

$$I_a : \{(B, A)|\Delta_3 > 0\},\ I_b : \{(B, A)|\Delta_3 < 0\}.$$

When $\varepsilon > 0$ and $D_1 < 0$, the bifurcation curve $\Delta_3(B, A) = 0$ divides the (B, A)—parameter plane into the two subregions (see Fig. 2):

$$II_a : \{(B, A)|\Delta_3 > 0\},\ II_b : \{(B, A)|\Delta_3 < 0\},$$

where $\Delta_3(B, A) = 0$ is made up of three parts: $L_{21}^{\pm} : A = f_1^{\pm}(B)$ and $L_{22} : A = f_2(B)$.

For $\varepsilon > 0$ and $D_1 \neq 0$, we have the bifurcations of phase portraits of Eq. (1.2) shown in Fig. 3.